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Digital Image Processing by the Two-Dimensional Discrete Fourier Transform Method

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DIGITAL IMAGE PROCESSING BY THE TWO-DIMENSIONAL DISCRETE FOURIER TRANSFORM METHOD

BY

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RESERACH REPORT

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1.0 IMAGE PROCESSING

A. Introduction

The objective of image processing is to perform some type of process on an image, such as a photographic transparency or X-ray, so that certain image qualities such as brightness and contrast are enhanced, while undesirable qualities such as blurriness and distortion are attenuated. In the past decade high-speed general-purpose digital computers have reached the stage of development where digital processing of images has become feasible. Formerly, image processing was performed almost entirely by optical means which required elaborate and accurate processing equipment, as well as excessive preparation and set-up time. In contrast, image processing by digital computer requires very little setup time, has increased accuracy, greater operational flexibility, and has the added advantage of performing nonlinear mathematical operations with relative ease.

The theory of image processing by digital computers can best be explained by first discussing the basic theory of optical image processing. Optical image processing is a technique in which two-dimensional image functions are illuminated by a light beam and processed by a system of lenses, photographic transparencies, and other optical elements. A typical optical image processing system is described by Andrews and Pratt [1]. This system which is comprised of lenses, photographic transparencies, and a light source is shown on Figure 1-1. A transparency of the original image in the x_1, y_1 plane with transmittance $f(x_1, y_1)$ is illuminated by a light beam so that the electric field amplitude of the light at the input plane is proportional to $f(x_1, y_1)$. The first spherical lens produces an image of the transparency in the

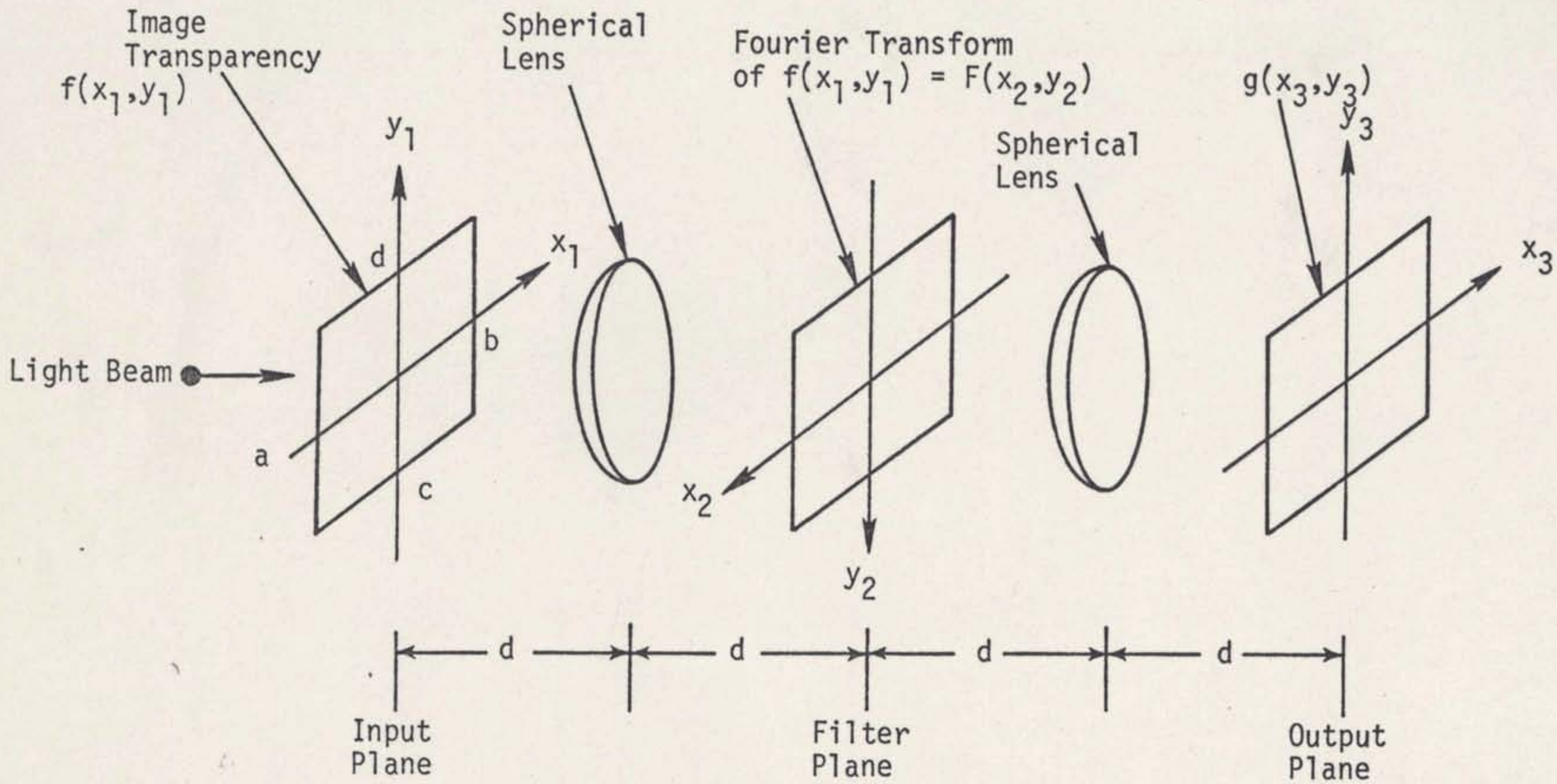

Figure 1-1. Optical Image Processing System

filter plane. The light electric field amplitude, $F(x_2, y_2)$, at the filter plane can be described by the two-dimensional Fourier transform relation

$$F(x_2,y_2) = \int_c^d \int_a^b f(x_1,y_1) e^{j\frac{2\pi}{\lambda d}(x_1 x_2 + y_1 y_2)}\, dx_1 dy_1 \qquad (1)$$

where λ is the wavelength of the light illuminating the transparency and d is the focal length of the lens. This relation can also be written as

$$F(u,v) = \int_c^d \int_a^b f(x_1,y_1) e^{j(ux_1+vy_1)}\, dx_1 dy_1 \qquad (2)$$

by defining u and v as

$$u = \frac{2\pi}{\lambda d} x_2 \text{ and } v = \frac{2\pi}{\lambda d} y_2 \qquad (3)$$

The variables u and v are called the spatial frequencies in the Fourier transform plane having dimensions of cycles per unit length. The relationship given by equation (2) then permits one to describe an image in terms of its spatial frequencies which result when a Fourier transform is made of the image function. A filter transparency with a transmittance function, $H(u,v)$, can be inserted at the filter plane to modify the amplitude and phase of the light there, and hence modify the spatial frequencies. A second spherical lens, as shown in Figure 1-1, will perform an inverse Fourier transform to return to the spatial domain. The electric field distribution, $g(x_3, y_3)$, in the output plane is then the inverse Fourier transform of the $H(u,v)$ $F(u,v)$ product and represents the processed image.

A major problem experienced with optical processing systems is the construction of filter transparencies. One technique is to let density variations on a transparency represent amplitude functions, and to let thickness variations represent phase functions. Unfortunately, variable thickness filters are rather difficult to construct. Other types of filter transparencies require relatively complex optical systems.

The operation performed by a lens in the optical image processing system can be performed mathematically by a digital computer since the operation can be described by a two-dimensional Fourier transform function. A block diagram of a digital image processing system is shown on Figure 1-2. The image to be processed is scanned by some type of scanner which converts the pictorial information to a two-dimensional continuous analog signal, f(x,y), which represents the gray scale variations of the image. The analog signal is then digitized by an analog to digital converter and stored. The digitized image data is represented by f(m,n) and is nothing more than a two-dimensional number array with elements proportional to the image spatial gray scale values. We will restrict ourselves to a square image so that the f(m,n) number array will always be an NxN array with the m and n integer denoting the array row and column numbers, respectively. A two-dimensional discrete Fourier transform of the f(m,n) array is represented by F(k,l), which is also an NxN array that represents the spatial frequency components of the image in the spatial frequency domain. Some type of mathematical operation is then performed in the frequency domain to alter the original image. This operation, denoted by H(k,l), can be a spatial frequency filter which alters the frequency and magnitude of the spatial frequencies, or it can be some other mathematical operation such as a data correlation operation. The ease with which a linear or nonlinear mathematical operation can be performed in the spacial frequency domain is what makes digital image processing so attractive. An inverse two-

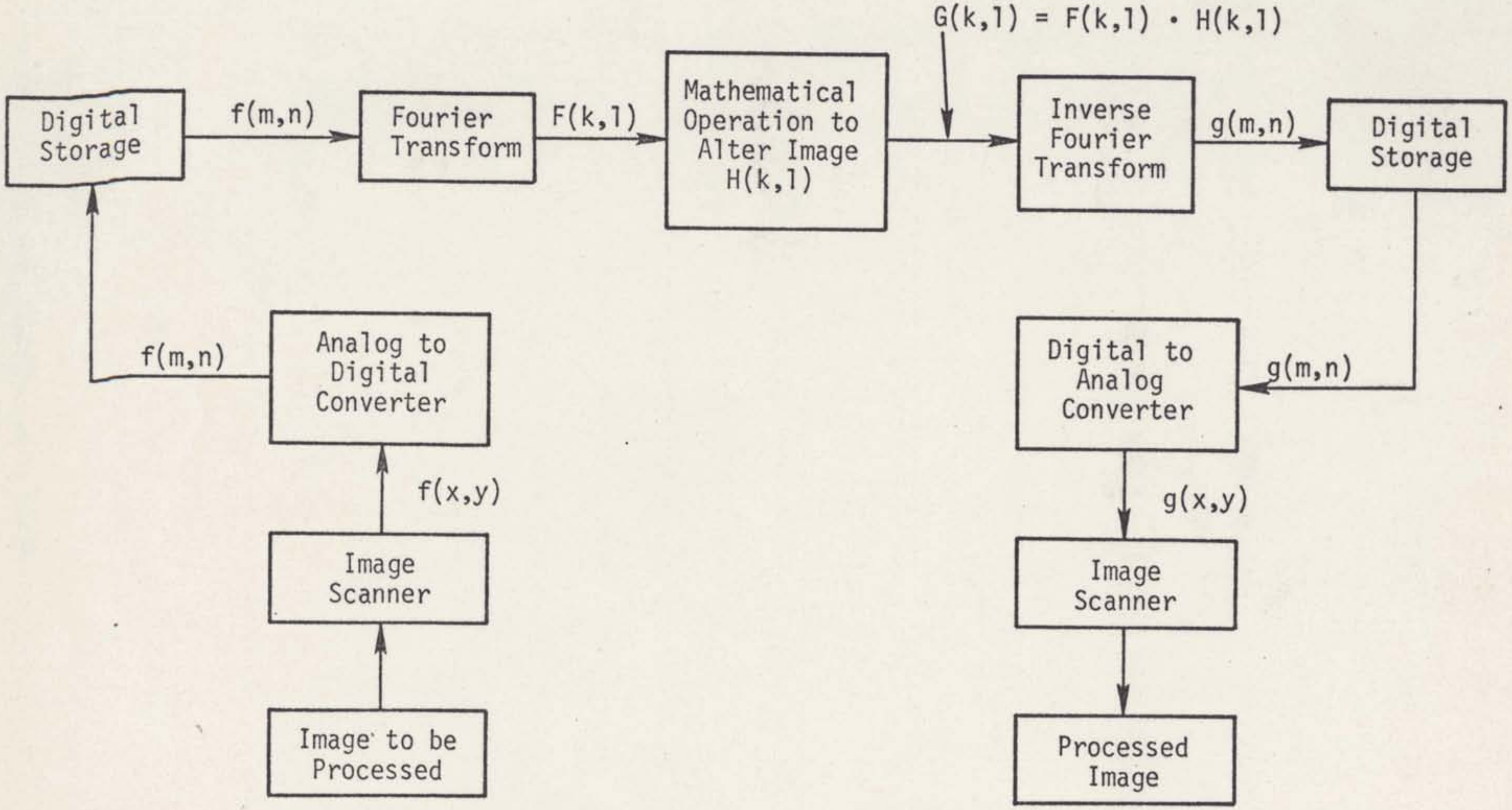

Figure 1-2. Digital Image Processing System

dimensional discrete Fourier transformation is performed and the digitized data of the altered image is denoted by $g(m,n)$. The digitized data is then converted back to an analog signal, $g(x,y)$, which is processed by the image scanner to create an improved image.

The original objective for this research paper was to be able to demonstrate how various types of image filters, represented by $H(k,l)$ in Figure 1-2, are employed in the processing of images. To achieve this objective required a thorough understanding of the theory and properties of Fourier transforms. Therefore, both one-dimensional and two-dimensional Fourier transforms, for both continuous and discrete functions, are discussed in later sections. The FORTRAN computer program that was written to perform a two-dimensional discrete Fourier transform is also included.

Recently, transform operations other than the Fourier transform have been developed. These transforms require less computer time and are better suited for certain mathematical image processing operations. A description of other transforms which includes the Walsh/Hadamard, Karhunen-Loeve, and Haar transforms is presented by Andrews [2].

B. Image Filtering Theory

The theory behind the use of two-dimensional Fourier transforms in image filtering is presented by Andrews, Tescher, and Kruger [3]. The theory is based on the assumption that if an object that is being photographed can be described as a two-dimensional function, $\phi(x,y)$, then the photographic image becomes

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta,x,y)\phi(\xi,\eta)d\xi d\eta \qquad (4)$$

where (ξ,η) are the spatial coordinates of the object of interest and

$g(\xi,\eta,x,y)$ is a function describing the effect of the optical devices on distorting the object in generating the image. The $g(\xi,\eta,x,y)$ function has often been referred to as the point-spread function or impulse response of the system, because the image would equal the point-spread function if a point source of light (a Dirac delta function) were the object of interest. If the point-spread function is invariant to shifts in the delta function, then

$$g(\xi,\eta,x,y) = g(x-\xi,y-\eta) \tag{5}$$

which suggests that the image is really a convolution of the object with the point-spread function. That is,

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\xi,\eta)g(x-\xi,y-\eta)d\xi d\eta \tag{6}$$

or

$$f(x,y) = \phi(x,y)*g(x,y) \tag{7}$$

where * denotes convolution.

The relationship given by equation (7) states that a distorted image $f(x,y)$ of an object is a result of the convolution of the object $\phi(x,y)$ with a function $g(x,y)$, which represents the image distortion. But convolution in the spatial domain is equivalent to multiplication in the spatial frequency domain. If u and v represent the spatial frequency components, then

$$F(u,v) = \Phi(u,v) \cdot G(u,v) \tag{8}$$

This relationship is the key to image filtering. Let $H(u,v)$ be a spatial frequency filter defined as

$$H(u,v) = 1/G(u,v) \tag{9}$$

Multiplying both sides of equation (8) by $H(u,v)$ results in the relationship

$$\Phi(u,v) = F(u,v) \cdot H(u,v) \tag{10}$$

Therefore, the original object can be recovered from the image by employing a spatial frequency filter, that represents the inverse of the distortion function, and then performing an inverse Fourier transform. In practice, only an approximation of the optimum spatial frequency filter can be determined because the exact characteristics of the distortion function $G(u,v)$ cannot be determined.

An image, as sensed by the retina or recorded on photographic film, can be considered as the product of two components: an illumination function and a reflectance function. These two components are considered to be independent of each other. The illumination function describes the illumination, or brightness, of a scene while the reflectance function describes the detail, or sharpness, of a scene. Thus, an image can be represented as a two-dimensional spatial function expressed in the form

$$I(x,y) = i(x,y)\ r(x,y) > 0 \tag{11}$$

where $I(x,y)$ is the image and $i(x,y)$ and $r(x,y)$ are the illumination and reflectance components, respectively. More detailed discussions of the relationship given by equation (11) are presented by Oppenheim, Schafer, and Stockham [4] and Gold and Rader [5]. The illumination function can be considered to encompass the lower spatial frequency components of an image while the reflectance function can be considered to encompass the higher spatial frequency components.

Two frequent objectives in image processing are dynamic-range compression and contrast enhancement. Dynamic-range compression is achieved by decreasing the light-to-dark ratios of an image. This can be achieved by a modified intensity $I'(x,y)$ related to $I(x,y)$ as

$$I'(x,y) = I^{\gamma}(x,y) \tag{12}$$

where γ is chosen to be positive but less than unity. This will decrease the amplitude of all the spatial frequency components and provide dynamic-range compression.

To enhance the contrast, or lend more sharpness to the edges of objects, the value of γ in equation (12) must be chosen to be greater than unity. This will increase the amplitude of all the spatial frequency components.

It is clear that dynamic-range compression and contrast enhancement are conflicting objectives. To achieve dynamic-range compression requires a γ less than unity while contrast enhancement requires a γ greater than unity. This can be rectified by separating the reflectance and illumination functions, modifying each with different gammas, and then recombining to form the modified image. The modified intensity is then given by

$$I''(x,y) = i^{\gamma i}(x,y)\ r^{\gamma r}(x,y) \tag{13}$$

where γ_i is less than unity to compress the dynamic range and γ_r is greater than unity to enhance the contrast. These gamma range selections will result in attenuation of the lower spatial frequency components and amplification of the higher spatial frequency components. Considering an image to be the product of an illumination function and a reflectance function, let the image be represented by the modified intensity function, that is,

$$f(x,y) = I''(x,y) \tag{14}$$

A linear or nonlinear filtering process can be performed on the image described by equation (13). The linear filtering process is known as homomorphic filtering and is discussed in detail by Gold and Rader

[5]. The homomorphic filter is directed toward separating the reflectance and illumination functions, performing a separate linear filtering operation on each component, and then recombining the components to obtain the processed image. To simplify notation, let the image be represented by

$$f_{xy} = i_{xy}^{\gamma_i} \, r_{xy}^{\gamma_r} \tag{15}$$

The illumination and reflectance components can be separated by a logarithm operation to get

$$l_{xy} = \ln f_{xy} = \gamma_i \ln i_{xy} + \gamma_r \ln r_{xy} \tag{16}$$

Let the logarithm functions be represented by

$$m_{xy} = \ln i_{xy} \text{ and } n_{xy} = \ln r_{xy} \tag{17}$$

so that equation (16) can be written as

$$l_{xy} = \gamma_i \, m_{xy} + \gamma_r \, n_{xy} \tag{18}$$

The equivalent of the above relationship in the spatial frequency domain is

$$L_{uv} = \gamma_i \, M_{uv} + \gamma_r \, N_{uv} \tag{19}$$

where u and v are the spatial frequency variables. The form of equation (19) permits a linear filtering operation to be performed in the spatial frequency domain where the values of γ_i and γ_r are controlled by the characteristics of the spatial frequency filter. A spatial frequency filter having the general shape shown in Figure 1-3 will attenuate the lower spatial frequencies and amplify the higher spatial frequencies. The complete homomorphic filtering operation is presented in Figure 1-4.

If a mathematical operation is not performed to separate the illumination and reflectance components prior to the filtering operation, then the filtering process is considered to be nonlinear. The nonlinear filtering operation is presented in Figure 1-5.

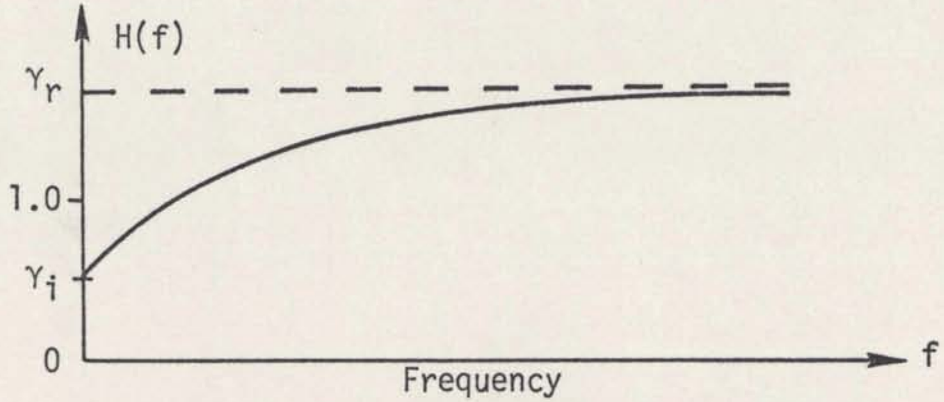

Figure 1-3. Spatial Frequency Filter

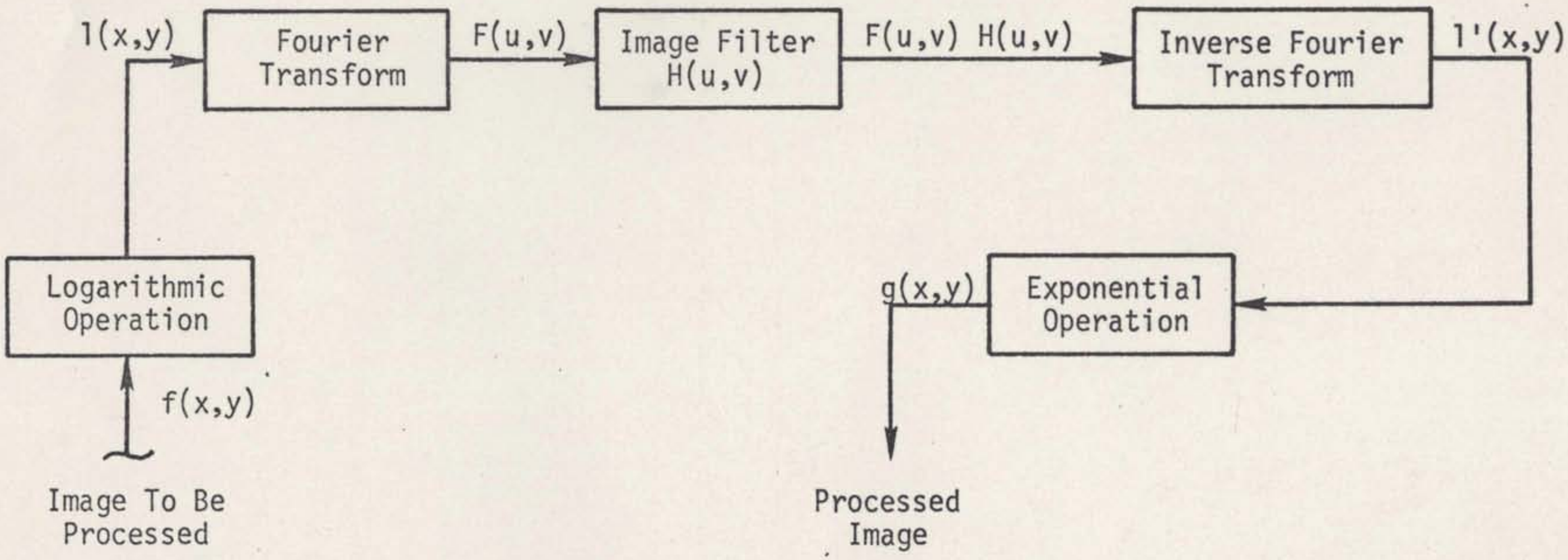

Figure 1-4. Linear Image Filtering System

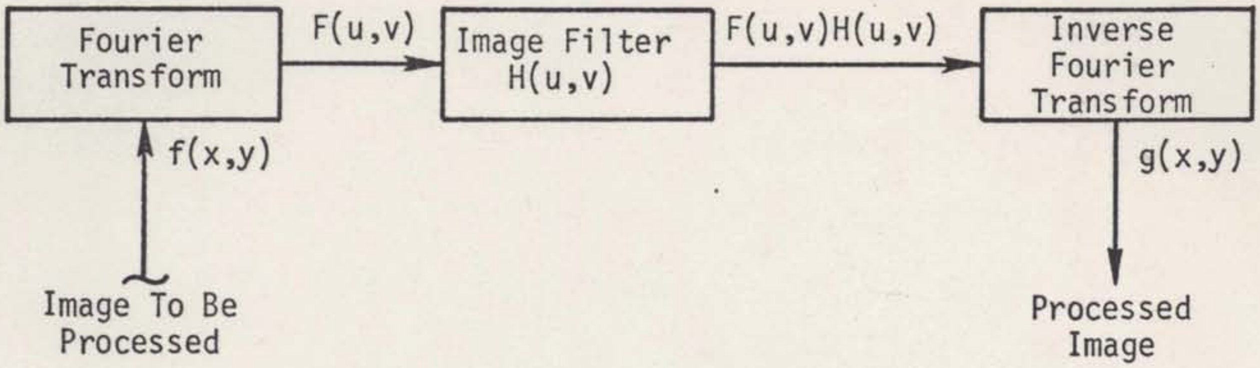

Figure 1-5. Nonlinear Image Filtering System

C. Examples of Linear and Nonlinear Spatial Filtering

Access to equipment necessary for scanning and digitizing the gray levels of actual photographic images was not available for the writing of this paper. Therefore, a simulated image consisting of nine gray code levels was devised to demonstrate the effects of different spatial image filters. The simulated image consists of the letters F, T, and U with gray levels of 8, 6, and 4, respectively. A horizontal line under the letters has a gray level of two and the image background has a gray level of unity. The 32x32 array representing the digitized gray code levels of the simulated image is shown on Figure 1-6. The two dimensional-discrete Fourier transform of the array is also a 32x32 array of complex numbers. As discussed in Section 3.0, half of the elements of the two-dimensional discrete Fourier transform array correspond to negative spatial frequency values and are the conjugates of the positive spatial frequency values. Further, these conjugate pairs and their absolute values are symmetrical about the zero-zero reference point in the spatial frequency domain. The absolute values of the complex array elements for one quadrant of the spatial frequency array are shown in Figure 1-7. Notice that the

```
         0 ------------ n ------------> 31

  0      11111111111111111111111111111111
  |      11111111111111111111111111111111
  |      11888888888888111111111111111111
  |      11888888888888111111111111111111
  |      11888888888888111111111111111111
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  |      11888888881116666111111111111111
  v      11888888881116666114444111444411
  m      11888888881116666114444111444411
  |      11888888881116666114444111444411
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  |      11111111111111111111111111111111
  |      11111111111111111111111111111111
  |      11222222222222222222222222222211
  |      11222222222222222222222222222211
  v      11111111111111111111111111111111
  31     11111111111111111111111111111111
```

Figure 1-6. Gray Level Array of Simulated Image l

K	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0:	3016	591	371	429	441	360	278	73	38	143	101	178	250	80	106	91
1:	551	397	294	295	89	250	94	46	23	80	24	99	74	50	42	23
2:	303	63	123	146	219	92	72	18	31	20	61	28	73	28	34	40
3:	197	224	170	50	90	61	90	20	23	35	63	23	2	11	57	30
4:	38	235	141	76	65	18	43	29	20	22	41	28	28	14	21	12
5:	205	138	27	60	38	50	4	23	16	13	13	30	26	20	11	12
6:	298	116	36	29	43	48	33	18	8	15	14	3	13	11	20	14
7:	15	30	59	76	23	17	18	12	5	1	2	22	17	4	11	11
8:	71	5	20	11	28	17	4	3	0	5	7	2	6	4	8	2
9:	83	52	46	19	39	30	24	4	5	6	6	14	12	13	8	0
10:	73	98	54	34	20	3	11	8	4	13	19	3	5	4	10	10
11:	125	67	25	25	33	3	15	2	11	3	9	17	21	6	4	2
12:	127	100	29	41	10	22	25	17	4	6	6	12	7	10	13	9
13:	158	96	54	31	12	4	15	14	10	15	7	11	3	5	7	11
14:	71	47	26	58	37	13	11	7	3	3	7	18	16	3	6	9
15:	9	23	14	13	6	14	10	7	4	2	2	7	2	7	7	5

Figure 1-7. Magnitude of One Quadrant of the Image Spatial Frequency Array

values are greater near the origin. This indicates that the lower spatial frequencies are predominant which is a characteristic of most images. The high value at the origin represents the average gray level of the image.

A shading plot program developed by the Martin Marietta Corporation Data Center at Orlando was used as a gray level image display device. The program is capable of displaying seven distinct gray shades on a cathode ray tube. Since nine gray shades were selected for the original image, all gray levels are proportionately decreased and truncated to integers for display purposes. The display of the original image is shown on Figure 1-8. The natural logarithm of the magnitude of the elements of the corresponding two-dimensional Fourier transform are shown on Figure 1-9. The logarithmic display of the elements compensates for the large magnitude variations and permits a seven-bit gray level display of the spatial frequency elements.

Four different types of spatial image filters were selected to demonstrate the nonlinear image filtering system shown in Figure 1-5. The characteristics of the four filters are shown on Figure 1-10. The spatial frequency filter is a function of the variable f where the variable is defined as

$$f = (k^2 + l^2)^{1/2} \tag{20}$$

The k and l variables represent the row and column numbers, respectively, of the two-dimensional Fourier transform array. The two most basic types of spatial filters are the low pass and high pass filters. The characteristics of the low pass filter are shown in Figure 1-10(a). This filter passes the average value and lower spatial frequency components and blocks the higher spatial frequencies. As mentioned

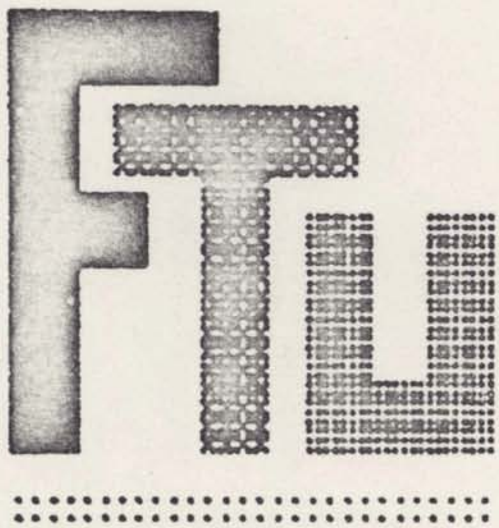


Figure 1-8. Original Image

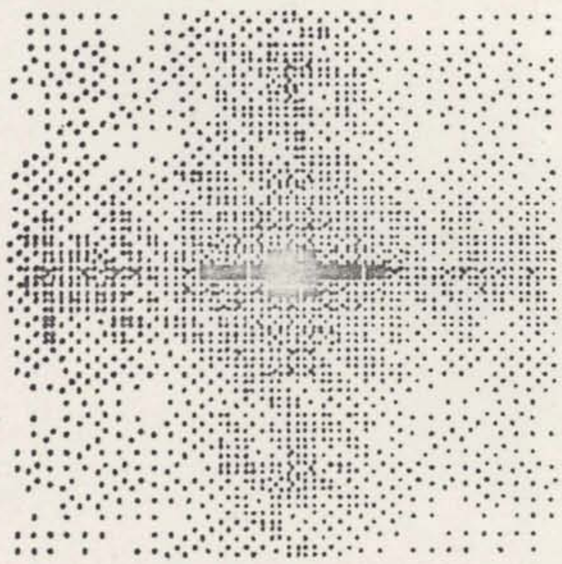


Figure 1-9. Logarithmic Display of Fourier Transform of Original Image

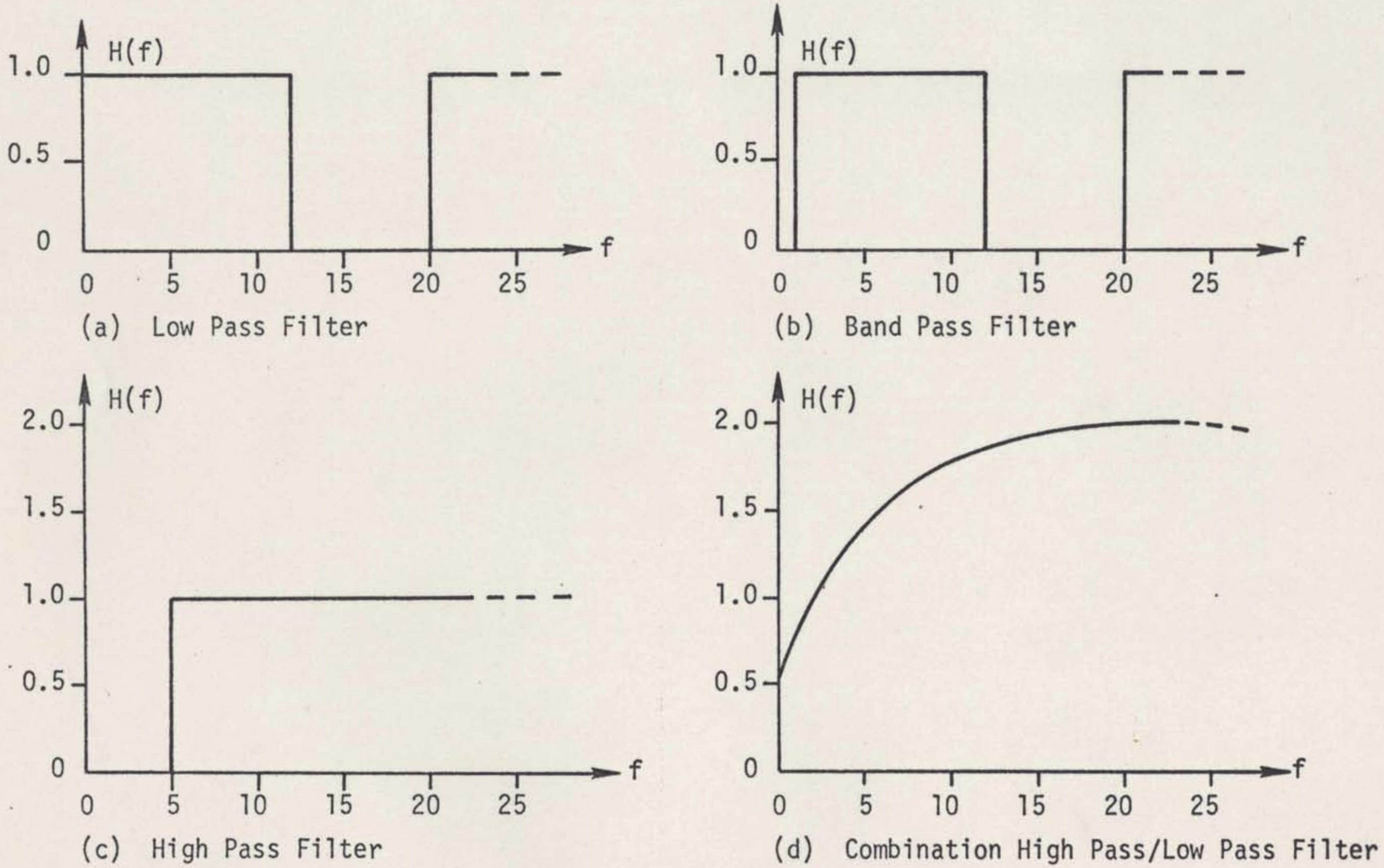


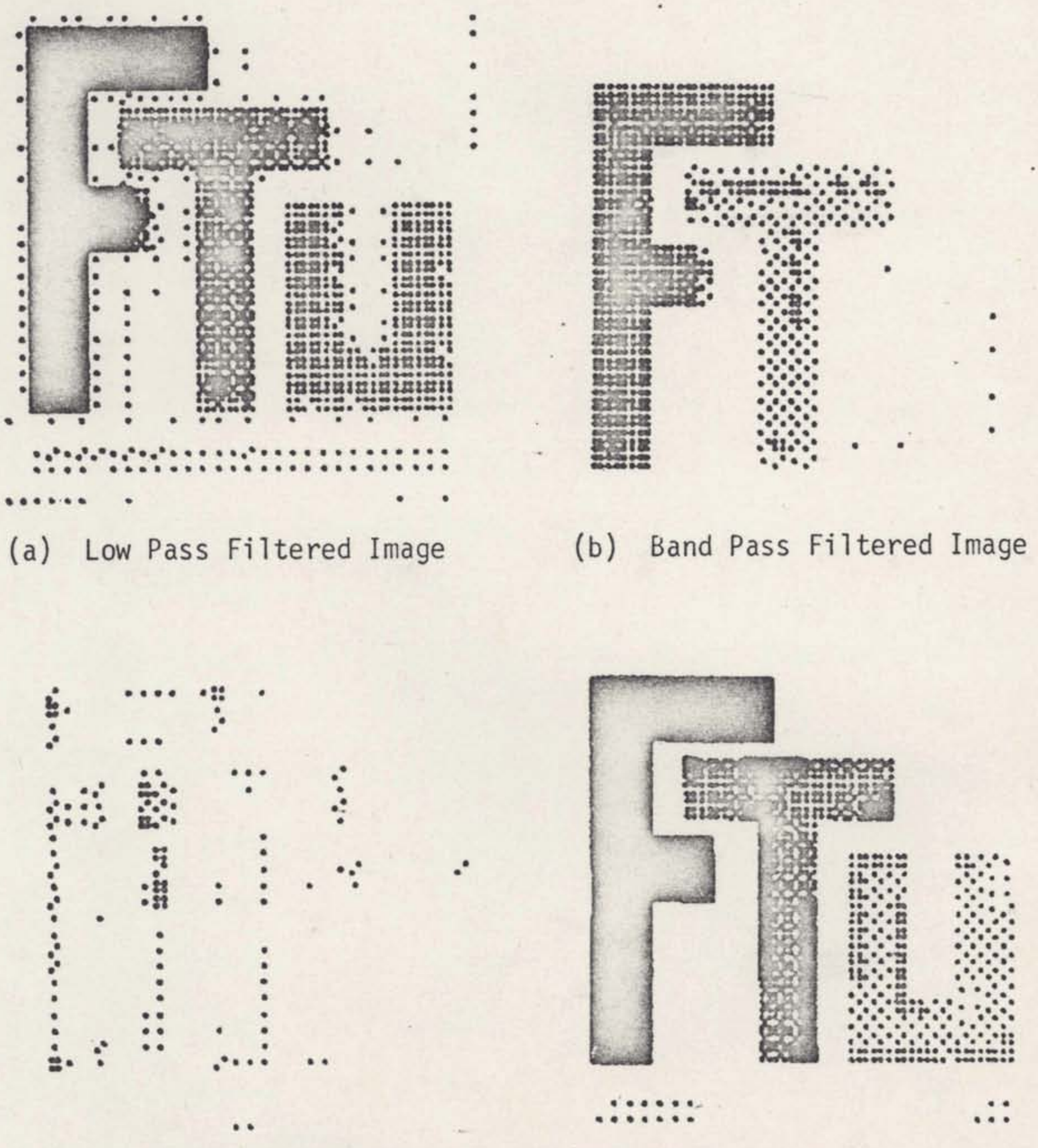
Figure 1-10. Spatial Frequency Filter Characteristics for Nonlinear Image Processing System

previously, the Fourier transform array consists of conjugate pairs which represent positive and negative spatial frequencies. Therefore, the filter is unity at f equal twenty so that the conjugate corresponding to the negative spatial frequency value is passed. This property will be true of all the spatial filters discussed. The resulting low pass filtered image is shown in Figure 1-11(a). The low pass filter has the tendency to make the edges less defined and increase the average gray shade level.

To illustrate the contribution of the average value component, which corresponds to f equal to zero, the low pass filter with a zero average value as shown by Figure 1-10(b) was tried. This filter has the characteristic of a band pass filter which can be designed to pass a selected band of spatial frequencies. The band pass filtered image is shown on Figure 1-11(b). Blocking or attenuating the average value component decreases the darkness of the image.

The high pass filter which passes only the higher spatial frequencies is presented in Figure 1-10(c). The corresponding high pass filtered image is shown in Figure 1-11(c). Only the edges of the letters are outlined which indicates that the high spatial frequency components are necessary to achieve sharpness of edges in an image. Processing of an X-ray with a high pass filter might disclose the outline of an object, such as a tumor, that was unnoticeable in the unfiltered image.

A filter which combines the characteristics of the low and high pass filters is presented in Figure 1-10(d). This filter attenuates the lower spatial frequencies and amplifies the higher spatial frequencies. This type of filter whitens the image and adds sharpness to



(a) Low Pass Filtered Image

(b) Band Pass Filtered Image

(c) High Pass Filtered Image

(d) High Pass/Low Pass Filtered Image

Figure 1-11. Nonlinear Spatial Filtered Images the edges. This type of filter results in the image shown in Figure 1-11(d).

Four different types of spatial image filters were selected to demonstrate the linear image filtering system depicted on Figure 1-4. The characteristics of the four filters are shown on Figure 1-12. All four curves attenuate at the lower spatial frequencies and amplify at the higher spatial frequencies. The curves have a peak amplitude at $f = 22.5$, which corresponds to k and l both being equal to 16 in equation (20). The amplitudes decrease at the higher frequencies to provide identical properties for the negative, or conjugate, spatial frequency components. The exponential filters increase exponentially over the full range of frequencies, where the S-filters are essentially constant at lower frequency values before increasing exponentially at the higher frequencies. These curves are representative of the many different types of spatial filters that can be selected for linear image filtering.

Application of the high gain exponential filter having the properties shown in Figure 1-12(a) results in the image shown in Figure 1-13(a). The results clearly show that the linear filtering technique is superior. The tendency is to add sharpness to the edges and lighten the overall image. Both dynamic-range compression and contrast enhancement are achieved. Application of the low gain exponential filter results in the image shown in Figure 1-12(b). Decreasing the gain whitens the image, yet retains the sharp edges.

The S-filter characteristics of Figure 1-12(b) provide darker edges and lighter overall images than the exponential filter. This is demonstrated by the image in Figure 1-13(c) which is a result of the

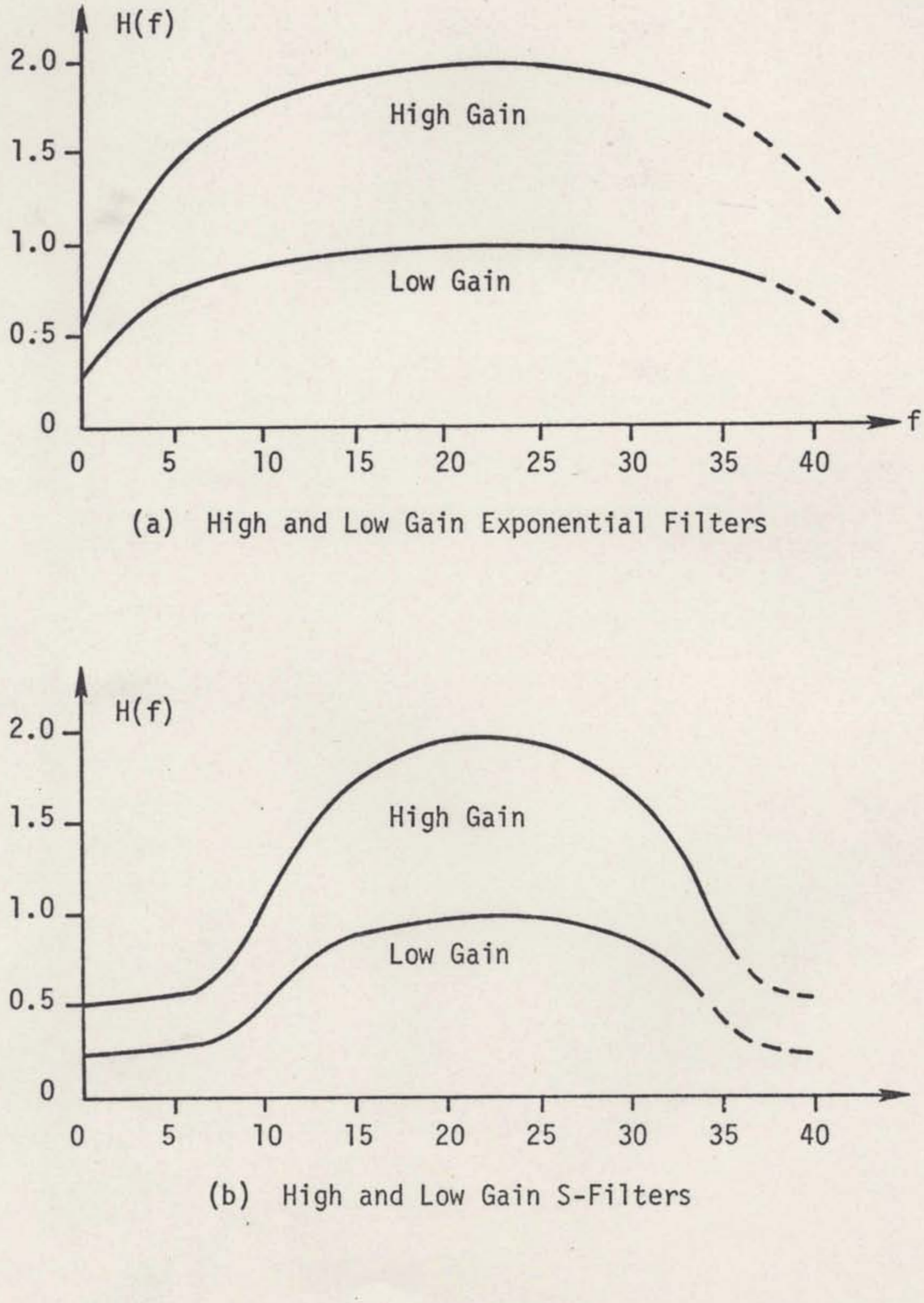
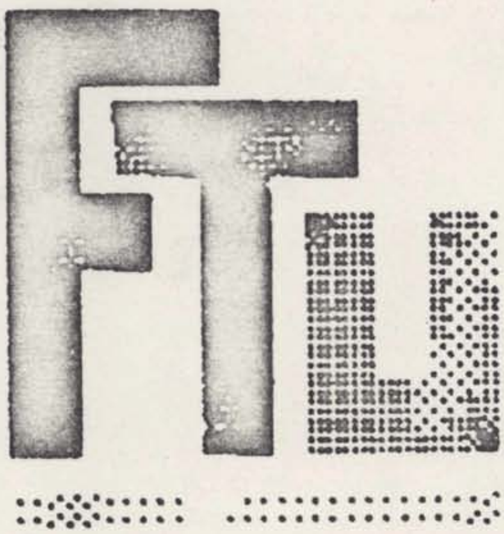
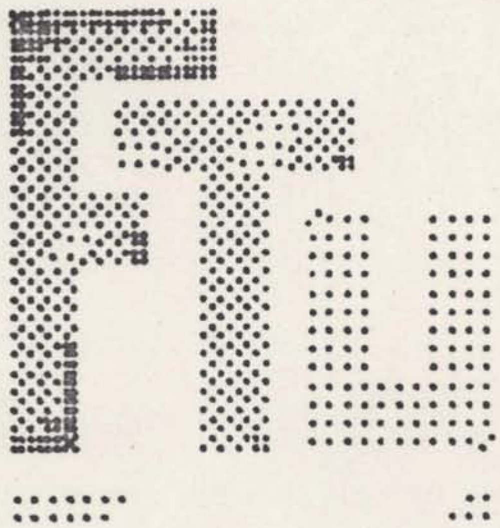


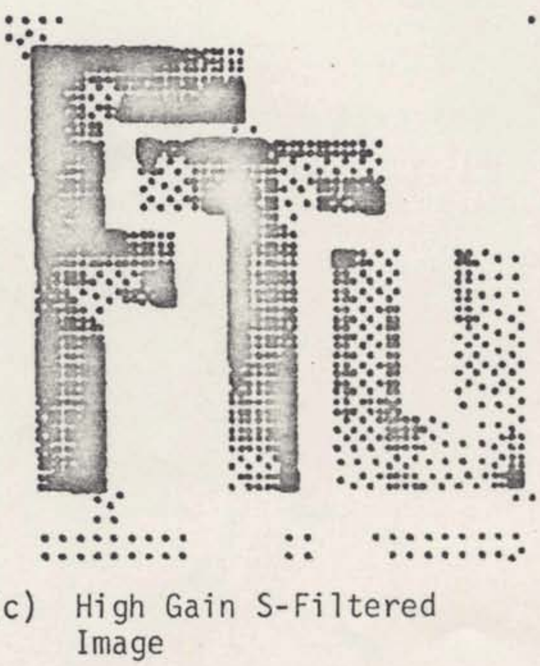
Figure 1-12. Spatial Frequency Filter Characteristics
for Linear Image Processing System



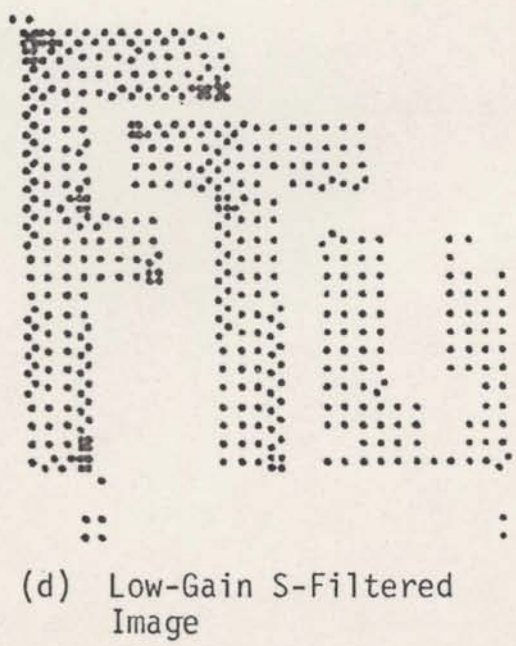
(a) High Gain Exponentially Filtered Image



(b) Low Gain Exponentially Filtered Image



(c) High Gain S-Filtered Image



(d) Low-Gain S-Filtered Image

Figure 1-13. Linear Spatial Filtered Images application of the high gain S-filter. Application of the low gain S-filter results in the image shown on Figure 1-13(d).

The preceding examples are based on an image represented by only a 32x32 number array and a seven gray shade display. Phenomenal image processing results using a 340x340 image array and twelve gray shades have been achieved by Oppenheim, Schafer, and Stockham [4]. Image processing techniques such as distribution linearization and image subtraction which do not employ a transformation of the digitized image data prove to be valuable in certain applications [6].

2.0 THE ONE-DIMENSIONAL FOURIER TRANSFORM

A. The Continuous One-Dimensional Fourier Transform

The Fourier transform of a continuous function g(t) is defined as

$$F(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \tag{1}$$

where both F(f) and g(t) may be complex functions. In most engineering applications the continuous function g(t) will be real and its Fourier transform complex. The information obtained from the Fourier transform indicates what waves of different frequencies must be additively combined to duplicate the function g(t). For each value of f the corresponding complex value of F(f) determines the relative magnitude and phase shift of each wave.

The ability of the Fourier transform to distinguish waves of different frequencies can best be explained by using the orthogonality property of the set of functions involved, namely, $\cos 2\pi m f_0 t$ and $\sin 2\pi n f_0 t$ where m and n take on integer values and f_0 is the first harmonic (fundamental) frequency. Let T be the period defined as $T = 1/f_0$. These cosine and sine functions are orthogonal over the interval from t_0 to t_0+T for any t_0. For simplicity the value $t_0 = 0$ will be used. The orthogonality property of the functions results in the following relationships.

$$\int_0^T \cos(2\pi m f_0 t)dt = 0, \text{ for all } m \neq 0 \tag{2}$$

$$\int_0^T \sin(2\pi n f_0 t)dt = 0, \text{ for all } n \tag{3}$$

$$\int_0^T \sin(2\pi n f_0 t)\cos(2\pi m f_0 t)dt = 0, \text{ for all } m \text{ and } n \tag{4}$$

$$\int_0^T \sin(2\pi m f_0 t)\sin(2\pi n f_0 t)dt = 0, \text{ for all } m \neq n \tag{5}$$

$$\int_0^T \cos(2\pi m f_0 t)\cos(2\pi n f_0 t)dt = 0, \text{ for all } m \neq n \tag{6}$$

In (5) and (6) a non-zero integral is obtained only when m and n are equal; thus

$$\int_0^T \cos^2(2\pi m f_0 t)dt = \frac{T}{2}, \text{ for all } m \tag{7}$$

$$\text{and } \int_0^T \sin^2(2\pi n f_0 t)dt = \frac{T}{2}, \text{ for all } n \tag{8}$$

The properties of (4) through (8) can now be used to arrive at some important relationships that always appear when applying the Fourier transform. Letting $\omega_0 = 2\pi f_0$, consider the following trigonometric identity:

$$e^{j(m\pm n)\omega_0 t} = (\cos m\omega_0 t \cos n\omega_0 t \mp \sin m\omega_0 t \sin n\omega_0 t)$$
$$\pm j(\cos m\omega_0 t \sin n\omega_0 t + \sin m\omega_0 t \cos n\omega_0 t) \tag{9}$$

Taking the integral of both sides of equation (9) and applying equations (4) through (8) the following relationships are obtained:

$$\int_0^T e^{j2\pi(m+n)f_0 t}dt = 0, \text{ for all m and n} \tag{10}$$

$$\int_0^T e^{j2\pi(m-n)f_0 t}dt = \begin{matrix} 0, \text{ for all } m \neq n \\ T, \text{ for } m = n \end{matrix} \tag{11}$$

Let g(t) in equation (1) be expressed as a Fourier series:

$$g(t) = \sum_{i=0}^{\infty} [A_i \cos 2\pi f_i t + B_i \sin 2\pi f_i t] \tag{12}$$

Now, g(t) will be expressed in exponential form, which is more convenient for application in equation (1).

$$g(t) = \sum_{i=0}^{\infty} A_i \left(\frac{e^{j2\pi f_i t} + e^{-j2\pi f_i t}}{2}\right) + \sum_{i=0}^{\infty} B_i \left(\frac{e^{j2\pi f_i t} - e^{-j2\pi f_i t}}{j2}\right) \tag{13}$$

$$g(t) = \sum_{i=0}^{\infty} \left(\frac{A_i - jB_i}{2}\right) e^{j2\pi f_i t} + \sum_{i=0}^{\infty} \left(\frac{A_i + jB_i}{2}\right) e^{-j2\pi f_i t} \tag{14}$$

To simplify this expression, we introduce a new coefficient to replace the A_i and B_i coefficients. By definition, let

$$C_i = \frac{A_i - jB_i}{2}, \; C_{-i} \frac{A_i + jB_i}{2}, \text{ and } C_0 = A_0 \tag{15}$$

The new form can now be written as

$$g(t) = \sum_{i=0}^{\infty} (C_i e^{j2\pi f_i t} + C_{-i} e^{-j2\pi f_i t}) = \sum_{i=-\infty}^{\infty} C_i e^{j2\pi f_i t} \tag{16}$$

Applying equation (1) the Fourier transform of g(t) becomes

$$F(f) = \int_{-\infty}^{\infty} \sum_{i=-\infty}^{\infty} C_i e^{j2\pi(f_i - f)t} dt = \int_{-\infty}^{\infty} \sum_{i=-\infty}^{\infty} C_i e^{j2\pi(f_i - f)t} dt \tag{17}$$

Several important properties of the Fourier transform can be determined from the above equation:

a) Comparison of (17) with (11) shows that the Fourier transform is non-zero only when $f = f_i$, and the value of the i^{th} frequency component is:

$$F(f_i) = C_i T \tag{18}$$

where T is the period or time span over which g(t) is defined.

b) The range of both t in the time domain and f in the frequency domain is from $-\infty$ to $+\infty$.

c) If g(t) is an even function then B_i in (15) will be zero and C_i will be real and equal to A_i. Therefore, the Fourier transform, $F(f_i)$ will be a real and even function. Similarly, if g(t) is an odd function A_i will be zero and $F(f_i)$ will be a complex and odd function.

d) If g(t) consists of both even and odd functions then C_i will have both real and imaginary components. As a result the Fourier transform will be complex and consist of both even and odd functions.

The Fourier transform is an invertible transform with the inverse given by

$$g(t) = \int_{-\infty}^{\infty} F(f)\, e^{j2\pi ft} df \tag{19}$$

By definition let the Fourier transform of g(t) be denoted by

$$F(f) = \overline{F}[g(t)] \tag{20}$$

and the inverse transform be denoted by

$$g(t) = \overline{F}^{\,-1}[F(f)] \tag{21}$$

A summary of other useful Fourier transform properties will now be given.

$$\overline{F}\,[f(-t)] = F^*(f) \tag{22}$$

$$\overline{F}\,[f(at)] = \frac{1}{a}\, F(\frac{f}{a}) \tag{23}$$

$$\overline{F}\,[af(t)] = a\; F(f) \tag{24}$$

$$\overline{F}\,[f_1(t)f_2(t)] = \int_{-\infty}^{\infty} F_1(\lambda)F_2(f-\lambda)d\lambda \tag{25}$$

$$\overline{F}^{\,-1}[F_1(\omega)F_2(\omega)] = \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau \tag{26}$$

$$\overline{F}\,[f(t-a)] = F(f)e^{-j2\pi af} \tag{27}$$

$$\overline{F}\,[f(t)e^{j2\pi f_0 t}] = F[(f-f_0)] \tag{28}$$

where * denotes the conjugate.

B. The Discrete One-Dimensional Fourier Transform

To compute discrete Fourier transforms of continuous functions with digital machines the functions are sampled and represented by a series of pulses. Only a finite number of samples are used to compute the transform. The time interval occupied by the finite set of samples automatically specifies the period of the fundamental frequency in the frequency domain while the number of samples taken in the time interval determine the range of frequencies or number of harmonics that can be detected. Let the finite sequence of N samples be represented by $f(nT)$ where $0 \leq n \leq N-1$ and T is the time interval between samples as shown in Figure 2-1.

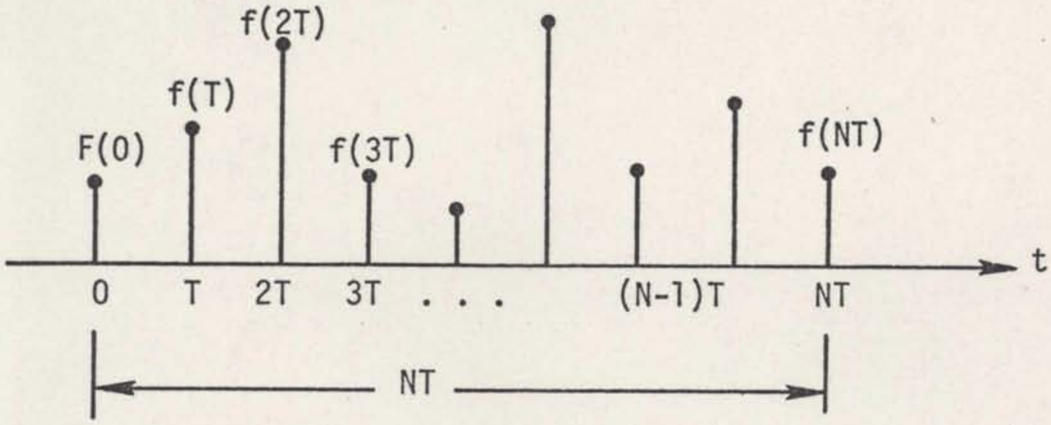


Figure 2-1. Finite Sampled Data Series

By definition the discrete Fourier transform is

$$F(kf_0) = \sum_{n=0}^{N-1} f(nT)\, e^{-j2\pi k f_0 nT}, \quad 0 \leq k \leq N-1 \tag{29}$$

where the period of the fundamental frequency component, f_0, in the frequency domain is equal to NT or

$$f_0 = \frac{1}{NT} \tag{30}$$

By substituting (30) into (29):

$$F(kf_0) = \sum_{n=0}^{N-1} f(nT) e^{\frac{-j2\pi kn}{N}} \tag{31}$$

There exists an inverse discrete Fourier transform which enables one to return to the time domain and is given by

$$f(nT) = \frac{1}{N} \sum_{K=0}^{N-1} F(kf_0) e^{+j\frac{2\pi kn}{N}}, \quad 0 \leq n \leq N-1 \tag{32}$$

The orthogonality properties which govern the continuous Fourier transform also apply to the discrete Fourier transform. If NT is the period corresponding to the fundamental frequency, f_0, the following relationships result, assuming p and q are greater or equal to zero:

$$\sum_{n=0}^{N-1} \cos(2\pi p f_0 nT) = \sum_{n=0}^{N-1} \cos(\frac{2\pi pn}{N}) = 0, \text{ for all } p \neq 0 \tag{33}$$

$$\sum_{n=0}^{N-1} \sin(\frac{2\pi qn}{N}) = 0, \text{ for all } q \tag{34}$$

$$\sum_{n=0}^{N-1} \sin(\frac{2\pi qn}{N}) \cos(\frac{2\pi pn}{N}) = 0, \text{ for all } p \text{ and } q \tag{35}$$

$$\sum_{n=0}^{N-1} \sin(\frac{2\pi qn}{N}) \sin(\frac{2\pi pn}{N}) = 0, \text{ for } p \neq q \tag{36}$$

$$\sum_{n=0}^{N-1} \cos(\frac{2\pi qn}{N}) \cos(\frac{2\pi pn}{N}) = 0, \text{ for } p \neq q \tag{37}$$

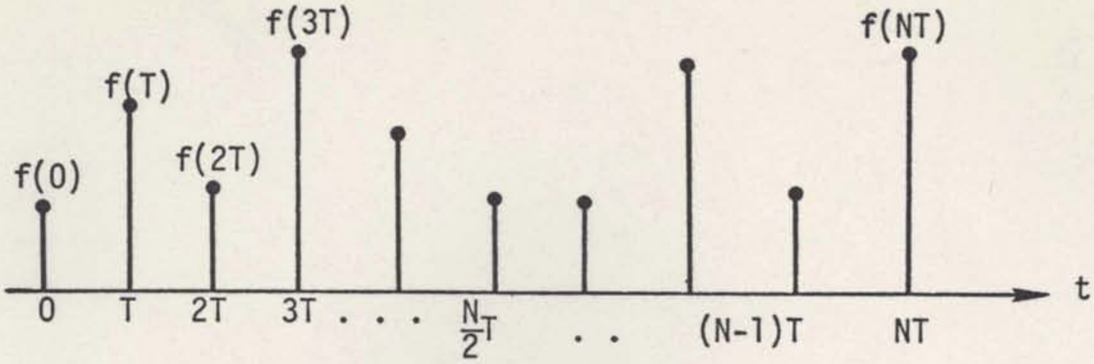
$$\sum_{n=0}^{N-1} \cos^2(\frac{2\pi pn}{N}) = \frac{N}{2}, \text{ for all } p \tag{38}$$

$$\sum_{n=0}^{N-1} \sin^2(\frac{2\pi qn}{N}) = \frac{N}{2}, \text{ for all } q$$

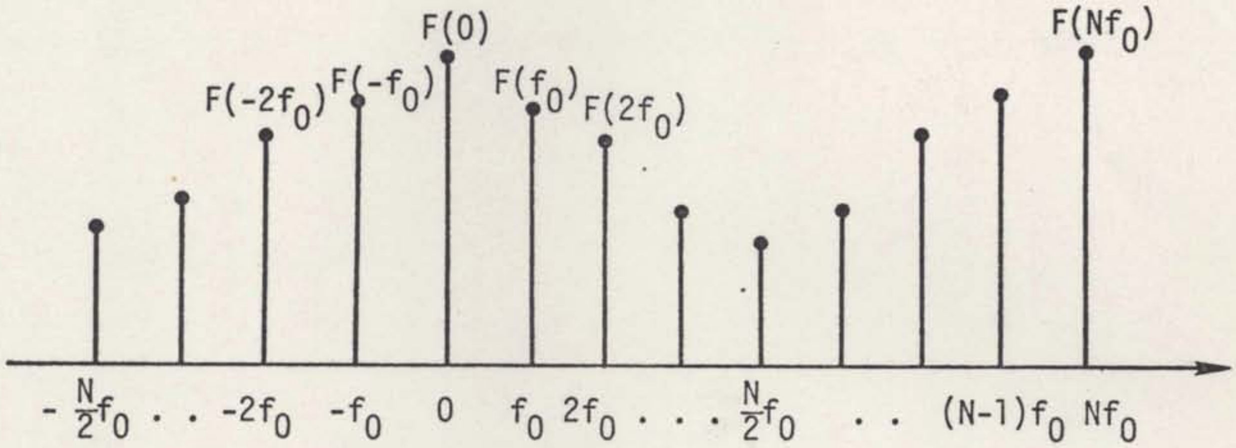
$$\sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}(q+p)} = 0, \text{ for all } q \text{ and } p \text{ greater than zero.} \tag{40}$$

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi n(q-p)}{N}} = \begin{matrix} 0, \text{ for } p \neq q \\ N, \text{ for } p = q \end{matrix} \tag{41}$$

The discrete Fourier transform has other characteristics which prove useful when switching back and forth between the frequency and time domains. Let $F(kf_0)$ represent the discrete Fourier transform of the time function f(nT) as shown in Figure 2-2.



(a) Time Domain



(b) Frequency Domain

Figure 2-2. Time Domain and Frequency Spectrum Notation

Applying the definition for the discrete Fourier transform given by (31)

$$F(0) = \sum_{n=0}^{N-1} f(nT) \tag{42}$$

$$F(\pm Nf_0) = \sum_{n=0}^{N-1} f(nT)\, e^{\pm j2\pi n} = \sum_{n=0}^{N-1} f(nT) = F(0) \tag{43}$$

Applying the definition for the inverse transform given by (32):

$$f(0) = \frac{1}{N}\sum_{k=0}^{N-1} F(kf_0) \tag{44}$$

$$f(\pm NT) = \frac{1}{N}\sum_{k=0}^{N-1} F(kf_0)\, e^{\pm j2\pi k} = \frac{1}{N}\sum_{k=0}^{N-1} f(kf_0) = f(0) \tag{45}$$

The relationships shown by equations (43) and (45) serve to show that both the frequency and time functions are assumed to be periodic with the period being determined by the number of samples, N.

Although N samples are used to compute the discrete Fourier transform only N/2 frequency components are obtained. The remaining N/2 frequency values correspond to negative frequency and time. To show this let k in equation (29) assume the value:

$$k = N-p, \text{ where } 0 \leq p \leq N/2 \tag{46}$$

To simplify notation let $F(kf_0)$ and $F(k)$ both represent the Fourier transform of a frequency component equal to k times the fundamental frequency, f_0. Also, in the time domain let $f(nT)$ and $f(n)$ be interchangeable and represent the same time value. The discrete Fourier transform of $k = N-p$ is:

$$F(N-p) = \sum_{n=0}^{N-1} f(n)e^{-j2\pi n}\, e^{+j\frac{2\pi np}{N}} = \sum_{n=0}^{N-1} f(n)e^{j\frac{2\pi np}{N}} \tag{47}$$

For k equal to $\pm p$:

$$F(-p) = \sum_{n=0}^{N-1} f(n)e^{j\frac{2\pi np}{N}} \tag{48}$$

$$F(p) = \sum_{n=0}^{N-1} f(n)\, e^{-j\frac{2\pi np}{N}} \tag{49}$$

Let $F^*(k)$ denote the conjugate of $F(k)$. Then from (47) through (49):

$$F(N-p) = F(-p) = F^*(p) \tag{50}$$

The above equation shows that the Fourier transform of identical frequency components which are opposite in sign are the conjugates of each other and that the Fourier transform values for $\frac{N}{2} < k < N$ represent values obtained for negative values of k. For example, let N and p equal 8 and 2 respectively. Then,

$$F(6) = F(-2) = F^*(2) \tag{51}$$

Frequency components corresponding to $k > \frac{N}{2}$ cannot be accurately distinguished according to the Sampling Theorem [7].

The sampling theorem states that, "if a function of time contains no frequencies higher than W Hertz, it is completely determined by giving the value of the function at a series of points spaced 1/2W seconds apart." Applying the theorem to our case indicates that in order to accurately detect any frequency components having a period less than twice the sampling interval T, (a frequency greater than $Nf_0/2$) would necessitate decreasing the time interval between samples. Thus, to detect higher frequencies requires higher sampling rates.

The discrete Fourier transform can be compactly written in matrix notation by letting

$$W = e^{-j\frac{2\pi}{N}} \tag{52}$$

Substituting this relationship into equations (31) and (32) we get

$$f(kf_0) = \sum_{n=0}^{N-1} f(nT)\, W^{kn} \tag{53}$$

$$\text{and } f(nT) = \frac{1}{N} \sum_{k=0}^{N-1} F(kf_0)\, W^{-kn} \tag{54}$$

To simplify notation let the following variables be interchangeable:

$$f(nT) = f_n \tag{55}$$

$$\text{and } F(kf_0) = F_k \tag{56}$$

Now, consider the following vectors:

$$\overline{f} = (f_0, f_1, f_2, \ldots, f_n, \ldots, f_{N-1}),\ 0 \leq n \leq N-1 \tag{57}$$

$$\overline{F} = (F_0, F_1, F_2, \ldots, F_k, \ldots, F_{k-1}),\ 0 \leq k \leq N-1 \tag{58}$$

$$\overline{W}_k = [W^0, W^k, W^{2k}, \ldots, W^{nk}, \ldots, W^{(n-1)k}],\ 0 \leq n \leq N-1 \tag{59}$$

$$\overline{W} = (\overline{W}_0, \overline{W}_1, \overline{W}_2, \ldots, \overline{W}_k, \ldots, \overline{W}_{(N-1)k}),\ 0 \leq k \leq N-1 \tag{60}$$

where $\overline{f}$ and $\overline{F}$ are now vectors which represent the discrete time and frequency values, respectively, and $\overline{W}$ is a row vector with the $\overline{W}_k$ column vectors as its elements. The discrete Fourier transform in matrix form is then:

$$\overline{F} = \overline{f} \cdot \overline{W} \tag{61}$$

For example, if N=4 the transform in matrix notation is:

$$(F_0, F_1, F_2, F_3) = (f_0, f_1, f_2, f_3) \cdot (\overline{W}_0, \overline{W}_1, \overline{W}_2, \overline{W}_3) \tag{62}$$

$$= (f_0, f_1, f_2, f_3) \cdot \begin{bmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{bmatrix} \tag{63}$$

$$= (f_0, f_1, f_2, f_3) \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j1 & -1 & +j1 \\ 1 & -1 & 1 & -1 \\ 1 & +j1 & -1 & -j1 \end{bmatrix} \tag{64}$$

The $\overline{W}$ matrix is an NxN matrix, and due to the relationship given by (52) the matrix is a symmetric matrix. That is, the matrix is symmetric about the main diagonal and the matrix is equal to its transpose.

The matrix equation for returning to the time domain from the frequency domain is:

$$\overline{f} = \frac{1}{N} (\overline{F} \cdot \overline{W}^{-1}) = \frac{1}{N} (\overline{F} \cdot \overline{W}) \tag{65}$$

where $\overline{W}^{-1}$ is the inverse of the $\overline{W}$ matrix. Due to the properties of the scalar, W, each element, a_{nk}, in the matrix, $\overline{W}$, is the conjugate of its corresponding element in the inverse matrix.

Therefore, if N=4 the time domain values are given by:

$$\overline{f} = (f_0, f_1, f_2, f_3) = \frac{1}{N} (F_0, F_1, F_2, F_3) \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j1 & -1 & -j1 \\ 1 & -1 & 1 & -1 \\ 1 & -j1 & -1 & j1 \end{bmatrix} \quad (66)$$

As a complete numerical example let $\overline{f}(nT)$ be the sum of two sine waves, $g(nT)$ and $h(nT)$, as shown in Figure 2-3(a), where N=8, T=1, and f_0=1/8. Using vector notation the numerical values are:

$$\overline{g} = (0, 0.707, 1, 0.707, 0, -0.707, -1, -0.707) \quad (67)$$

$$\overline{h} = (0, 1, 0, -1, 0, 1, 0, -1) \quad (68)$$

$$\overline{f} = \overline{g}+\overline{h} = (0, 1.707, 1, -0.293, 0, 0.392, -1, -1.707) \quad (69)$$

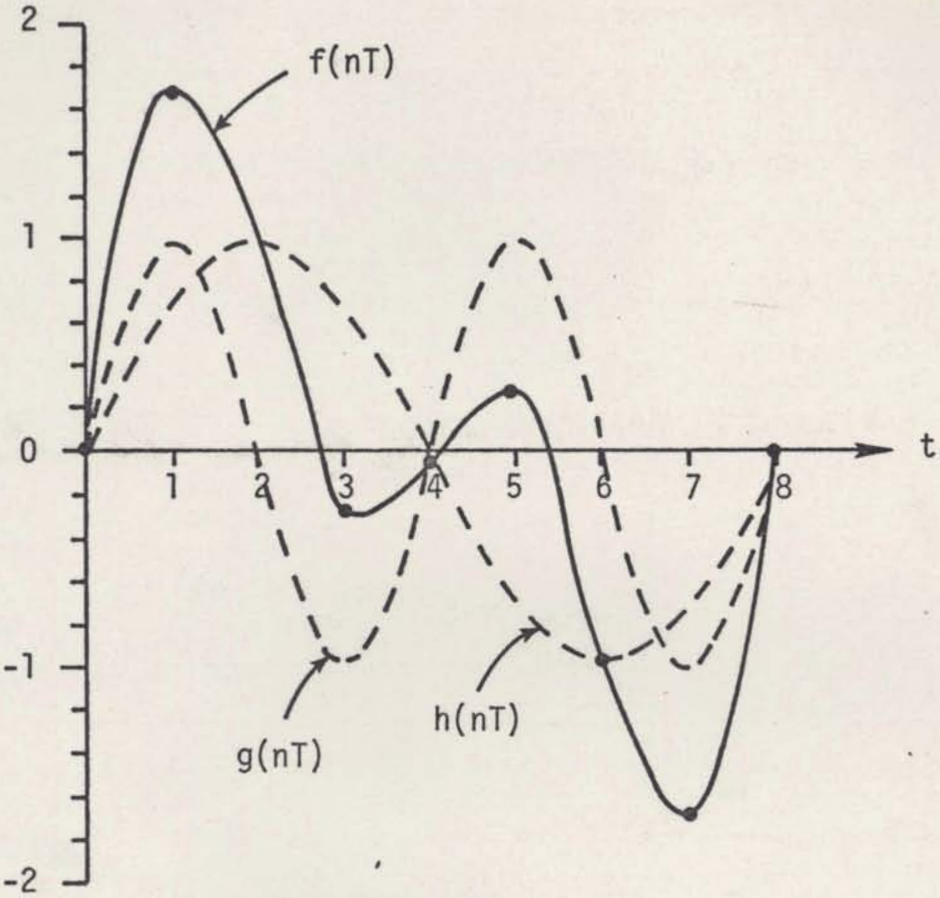
$$\overline{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & S(1-j1) & -j1 & S(-1-j1) & -1 & -S(1-j1) & j1 & S(1+j1) \\ 1 & -j1 & -1 & j1 & 1 & -j1 & -1 & j1 \\ 1 & S(-1-j1) & j1 & S(1-j1) & -1 & S(1+j1) & -j1 & S(-1+j1) \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & S(-1+j1) & -j1 & S(1+j1) & -1 & S(1-j1) & j1 & S(-1-j1) \\ 1 & j1 & -1 & -j1 & 1 & j1 & -1 & -j1 \\ 1 & S(1+j1) & j1 & S(-1+j1) & -1 & S(-1-j1) & -j1 & S(1-j1) \end{bmatrix} \quad (70)$$

where $S = 1/\sqrt{2}$

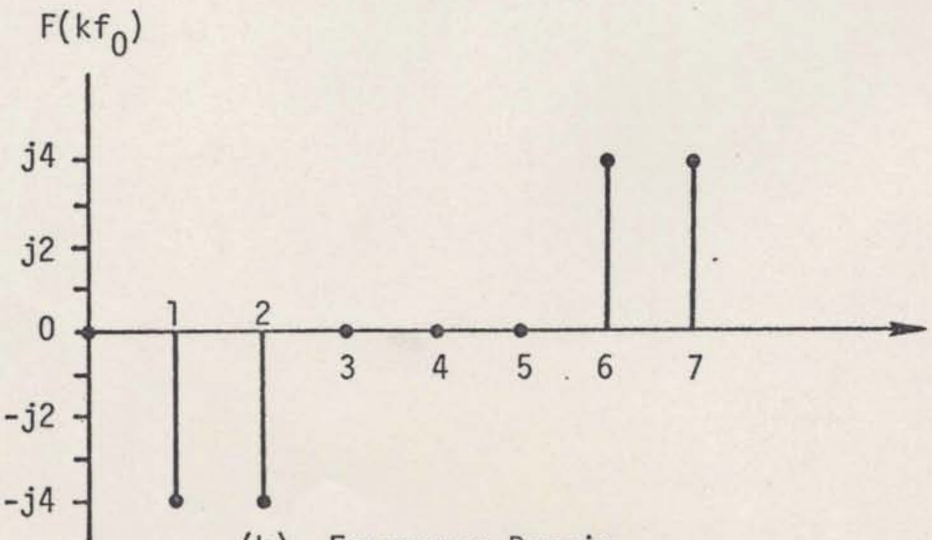
The discrete Fourier transform of f(nT) is:

$$\overline{F} = \overline{f} \cdot \overline{W} = (0, -j4, -j4, 0, 0, 0, j4, j4) \quad (71)$$

Notice that F_1 and F_2 are the conjugates of F_6 and F_7, respectively, and their magnitudes are equal to N/2. The resulting frequency spectrum is shown in Figure 2-3(b).



(a) Time Domain



(b) Frequency Domain

Figure 2-3. Discrete Fourier Transform of Function Consisting of the Sum of Two Sinusoids

3.0 THE TWO-DIMENSIONAL FOURIER TRANSFORM

A. The Continuous Two-Dimensional Fourier Transform

Where the one-dimensional Fourier transform distinguishes sinusoidal waves of different frequencies that additively combine to form a function in one-dimensional space, the two-dimensional Fourier transform (TDFT) distinguishes sets of two-dimensional sinusoidal corrugated surfaces which can be additively combined to form a function in two-dimensional space.

Let $f(x,y)$ be a two-dimensional continuous function and $F(u,v)$ be the resulting TDFT of the function. The relationship between the two is:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy \tag{1}$$

$$f(x,y) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv \tag{2}$$

Since these equations contain the function $\exp\,[j2\pi(ux+vy)]$, they can be separated into real cosine terms and imaginary sine terms. For explanation purposes consider the cosine component, remembering that the sine component is identical, except that it is in the imaginary domain. If $f(x,y)$ represents the height or magnitude at each point on a two-dimensional surface and a section is made through the surface in the x direction, the observed peaks and valleys will undulate with a frequency of u cycles per unit x. Similarly, a section made in the y direction will undulate with a frequency of v cycles per unit y. Whatever dimensions x and y may have, u and v will have the dimensions of cycles per unit x or y. The function $\cos[2\pi(ux+vy)]$ represents a cosinusoidally corrugated two-dimensional surface whose contours of constant height or magnitude coincide with lines whose equation is:

$$ux + vy = \text{constant} \tag{3}$$

The frequency domain function F(u,v) therefore, distinguishes what sinusoidal corrugations with frequencies of u cycles per unit x in the x direction and v cycles per unit y in the y direction are required to produce the f(x,y) function. When both the real and imaginary components of F(u,v) are included the proper phase and amplitude of each corrugation is obtained. Summation of the corrugations then results in the original two-dimensional surface, f(x,y).

Another way of looking at the TDFT is that a one-dimensional Fourier transform is made along the x axis for constant values of y to obtain the function g(u,y). Then, a transform is made along the y axis for constant values of u to obtain the F(u,v) function.

The two-dimensional Fourier transform has properties which parallel the one-dimensional Fourier transform. If we let $\overline{F}[f(t)]$ denote the Fourier transform of f(t) these properties are:

$$\overline{F}[f(ax,by)] = \frac{1}{|ab|} F(\frac{u}{a}, \frac{v}{b}) \tag{4}$$

$$\overline{F}[f(x,y) + g(x,y)] = F(u,v) + G(u,v) \tag{5}$$

$$\overline{F}[f(x,y) * g(x,y)] = F(u,v)\ G(u,v) \tag{6}$$

where * denotes convolution.

$$\overline{F}[f(x,y) * \overline{f}(-x,-y) = |F(u,v)|^2 \tag{7}$$

where $\overline{f}$ denotes conjugate.

B. The Discrete Two-Dimensional Fourier Transform

Let f(m,n) represent the sampled data of a two-dimensional continuous function, f(y,x) where the integers m and n correspond to the sample numbers along the y and x axes, respectively. Also, let the corresponding two-dimensional discrete Fourier transform (TDDFT) of f(m,n) be denoted as F(k,l). We will make the restriction that both functions can be described by an NxN square matrix of real or complex numbers where k, l, m, and n take on integer values. The subscripts k and l denote the row and column, respectively, of the F(k,l) array and m and n denote the row and column, respectively of the f(m,n) array as illustrated in Figure 3-1. The relationship between the two functions is

$$F(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) e^{-j\frac{2\pi}{N}(km+ln)} \tag{8}$$

$$f(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F(k,l) e^{j\frac{2\pi}{N}(km+ln)} \tag{9}$$

To simplify notation F(k,l) and f(m,n) will be used interchangeably with F_{kl} and f_{mn}, respectively. The k and l variables denote frequency multiplies of a fundamental frequency when the TDDFT is taken of the f(m,n) function.

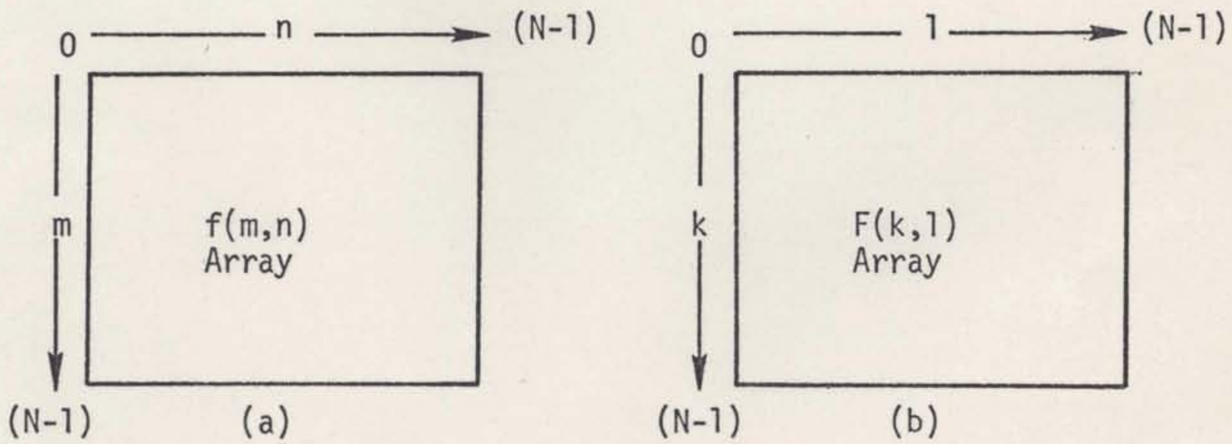


Figure 3-1. Two-Dimensional Array Notation

The frequency and period relationship of the TDDFT can be shown by rewriting equation (8) in a different form to include the fundamental frequency variable, f_0, and the time interval between samples, T. Including the f_0 and T variables the equation becomes

$$F(kf_0,lf_0) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(mT,nT)e^{-j2\pi(kf_0m + lf_0n)T} \tag{10}$$

Because we restricted ourselves to an NxN array the fundamental period in both dimensions is NT and the fundamental frequency is

$$f_0 = \frac{1}{NT} \tag{11}$$

When the above relationship is substituted into equation (10) the sampling interval variable, T, is eliminated which results in equation (8). The variable arguments for F(k,l) and f(m,n) in equation (11) were written as $F(kf_0,lf_0)$ and f(mT,nT) to stress that the k and l variables represent frequency multiples of f_0 and that the m and n variables represent sampled discrete values.

To explain the mechanics of the TDDFT let f(m,n) represent the sum of a complex sinusoid of frequency, p, in the y direction and of frequency, q, in the x direction, so that

$$f(m,n) = C_pe^{j\frac{2\pi}{N}pm} + C_qe^{j\frac{2\pi}{N}qn} \tag{12}$$

where C_p and C_q can be complex amplitudes. The TDDFT of F(m,n) becomes

$$F(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} C_pe^{j\frac{2\pi}{N}[(p-k)m-ln]} + \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} C_qe^{j\frac{2\pi}{N}[(q-l)n-km]} \tag{13}$$

Both terms of the above equation have the same form. Therefore, let's examine the first term on the right side by rewriting the term as,

$$G_p = C_p \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \{\cos\frac{2\pi}{N}[(p-k)m-ln] + j\sin\frac{2\pi}{N}[(p-k)m-ln]\} \tag{14}$$

Now, using the trigonometric identities,

$$\sin(x\pm y) = \sin x \cos y \pm \cos x \sin y \tag{15}$$

and $$\cos(x\pm y) = \cos x \cos y \pm \sin x \sin y \tag{16}$$

we can rewrite (14) as

$$G_p = C_p \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \{\cos[\tfrac{2\pi}{N}(p-k)m] \cos[\tfrac{2\pi}{N}ln] + \sin[\tfrac{2\pi}{N}(p-k)m] \sin[\tfrac{2\pi}{N}ln]\}$$

$$+ jC_p \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \{\sin[\tfrac{2\pi}{N}(p-k)m] \cos[\tfrac{2\pi}{N}ln] - \cos[\tfrac{2\pi}{N}(p-k)m] \sin[\tfrac{2\pi}{N}ln]\} \quad (17)$$

Assume that we are holding n constant and iterating the m variable from 0 to N-1. During this summing operation the sinusoidal terms having the ln variables are constant and equation (17) for the iteration of m can be expressed as

$$G_p = C_p \sum_{m=0}^{N-1} \{K_1 \cos[\tfrac{2\pi}{N}(p-k)m] + K_2 \sin[\tfrac{2\pi}{N}(p-k)m]\} \quad (18)$$

$$+ jC_p \sum_{m=0}^{N-1} \{K_1 \sin[\tfrac{2\pi}{N}(p-k)m] - K_2 \cos[\tfrac{2\pi}{N}(p-k)m]\}$$

where K_1 and K_2 are constants. We know from the orthogonality properties of the sine and cosine functions that

$$\sum_{m=0}^{N-1} \cos[\tfrac{2\pi}{N}(p-k)m] = \begin{cases} 0 \text{ for } p \neq k \\ 1 \text{ for } p = k \end{cases} \quad (19)$$

$$\text{and } \sum_{m=0}^{N-1} \sin[\tfrac{2\pi}{N}(p-k)m] = 0 \quad (20)$$

Applying the above relationships to equation (18) we see that the only condition for which G_p will be non-zero is when p equals k. Similarly, q must equal l when n is being iterated for the second term of equation (13) to be non-zero. Thus, the TDDFT is able to distinguish discrete frequencies in two dimensions.

Other characteristic properties of the TDDFT are best explained by employing matrix notation. By definition, let the scalar

$$W = e^{-j\frac{2\pi}{N}}$$

so that equation (8) can be written as

$$F_{kl} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f_{mn}\, W^{km}\, W^{ln} = \sum_{m=0}^{N-1} W^{km} \sum_{n=0}^{N-1} f_{mn}\, W^{ln} \tag{21}$$

Now, let the following matrices be defined as:

$$\overline{F} = \begin{bmatrix} F_{00} & \cdot\;\cdot\;\cdot & F_{01} & \cdot\;\cdot\;\cdot & F_{0(N-1)} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ F_{k0} & \cdot\;\cdot\;\cdot & F_{kl} & \cdot\;\cdot\;\cdot & F_{k(n-1)} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ F_{(n-1)0} & \cdot\;\cdot\;\cdot & F_{(N-1)l} & \cdot\;\cdot\;\cdot & F_{(N-1)(N-1)} \end{bmatrix} \tag{22}$$

$$\overline{f} = \begin{bmatrix} F_{00} & \cdot\;\cdot\;\cdot & F_{0n} & \cdot\;\cdot\;\cdot & F_{0(N-1)} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ f_{m0} & \cdot\;\cdot\;\cdot & f_{mn} & \cdot\;\cdot\;\cdot & f_{m(N-1)} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ f_{(N-1)0} & \cdot\;\cdot\;\cdot & f_{(N-1)n} & \cdot\;\cdot\;\cdot & f_{(N-1)(N-1)} \end{bmatrix} \tag{23}$$

$$\overline{W}^{\,ln} = \begin{bmatrix} W^0 & W^0 & W^0 & \cdots & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & \cdots & W^n & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & \cdots & W^{2n} & \cdots & W^{2N-2} \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ W^0 & W^1 & W^{2l} & \cdots & W^{ln} & \cdots & W^{l(N-1)} \\ \cdot & & & & \cdot & & \\ \cdot & & & & \cdot & & \\ \cdot & & & & \cdot & & \\ W^0 & & & & W^{(N-1)n} & \cdots & W^{2(N-1)} \end{bmatrix} \tag{24}$$

Since the variables k, l, m, and n are all in the range from zero to N-1, the $\overline{W}^{\,km}$ matrix for the km product is identical to that for the ln product. Therefore,

$$\overline{W} = \overline{W}^{\,ln} = \overline{W}^{\,km} \tag{25}$$

The matrix equation form of the TDDFT can then be written as:

$$\overline{F} = \overline{W}^{\,km} \cdot \overline{f} \cdot \overline{W}^{\,ln} = \overline{W} \cdot \overline{f} \cdot \overline{W} \tag{26}$$

The inverse transform of the above equation is

$$\overline{f} = \overline{W}^{\,-1} \cdot \overline{F} \cdot \overline{W}^{\,-1} \tag{27}$$

where $\overline{W}^{\,-1}$ is the inverse of the matrix, $\overline{W}$. As was discussed in Section 2.0 each element a_{nk} in the matrix, $\overline{W}$, is the conjugate of its corresponding element in its inverse matrix. Therefore, let the inverse matrix be represented by $\overline{W}^*$ so that equation (27) can be written as

$$\overline{f} = \overline{W}^* \cdot \overline{F} \cdot \overline{W}^*$$

If we define the matrix, Z, as

$$\overline{Z} = \overline{f} \cdot \overline{W} \tag{28}$$

then $\overline{Z}$ is the matrix formed by taking the one-dimensional Fourier transform of each row of the f(m,n) array, and the TDDFT is then determined by the relationship

$$\overline{F} = \overline{W} \cdot \overline{Z} \tag{29}$$

Similarly, the inverse transform is given by

$$\overline{f} = \frac{1}{N^2} (\overline{W}^* \cdot \overline{Z}^*) \tag{30}$$

where $\overline{Z}^* = \overline{F} \cdot \overline{W}^*$ (31)

The computer subroutine TWOFFT which will be discussed in Section 4.0 was written to compute the TDDFT by twice performing a one-dimensional discrete Fourier transform by the n log n method, which is also known as the fast Fourier transform method [8]. To perform a fast Fourier transform the matrix multiplication must be in the same form and order as shown by equation (28). The method by which the TWOFFT subroutine computes the TDDFT is as follows:

1 Perform a one-dimensional fast Fourier transform to determine the matrix $\overline{Z}$ as indicated by equation (28).

2 Take the transpose of $\overline{Z}$ to obtain

$$\overline{Z}^T = \overline{W}^T \cdot \overline{f}^T = \overline{W} \cdot \overline{f}^T \tag{32}$$

3 Perform a second one-dimensional fast Fourier transform on the transpose of $\overline{Z}$ and the result is the transpose of the TDDFT array. That is,

$$\overline{F}^T = \overline{Z}^T \cdot \overline{W} \tag{33}$$

4 The TDDFT array is thus obtained by taking the transpose of the above relationship to obtain

$$\overline{F} = \overline{W}^T \cdot \overline{Z} = \overline{W} \cdot \overline{f} \cdot \overline{W} \tag{34}$$

The TWOFFT will also perform an inverse transform to recover the original array by employing the $\overline{W}$* matrix in place of the $\overline{W}$ matrix. The actual TWOFFT subroutine is presented in Section 4.0.

The relationship between the elements of the $\overline{f}$ and $\overline{F}$ arrays can best be demonstrated by a numerical example which was used to check out the TWOFFT subroutine.

Let f(m) and f(n) be one-dimensional functions that vary along the y and x axes, respectively, and each represented by a series of eight samples as shown in Figure 3-2. The function f(m) represents a single cosinusoidal wave where f(n) represents the sum of a constant or dc component and two sinusoidal waves. The two functions are then defined by

$$f(m) = \cos\frac{2\pi m}{N} = \cos\frac{\pi}{4}m \tag{35}$$

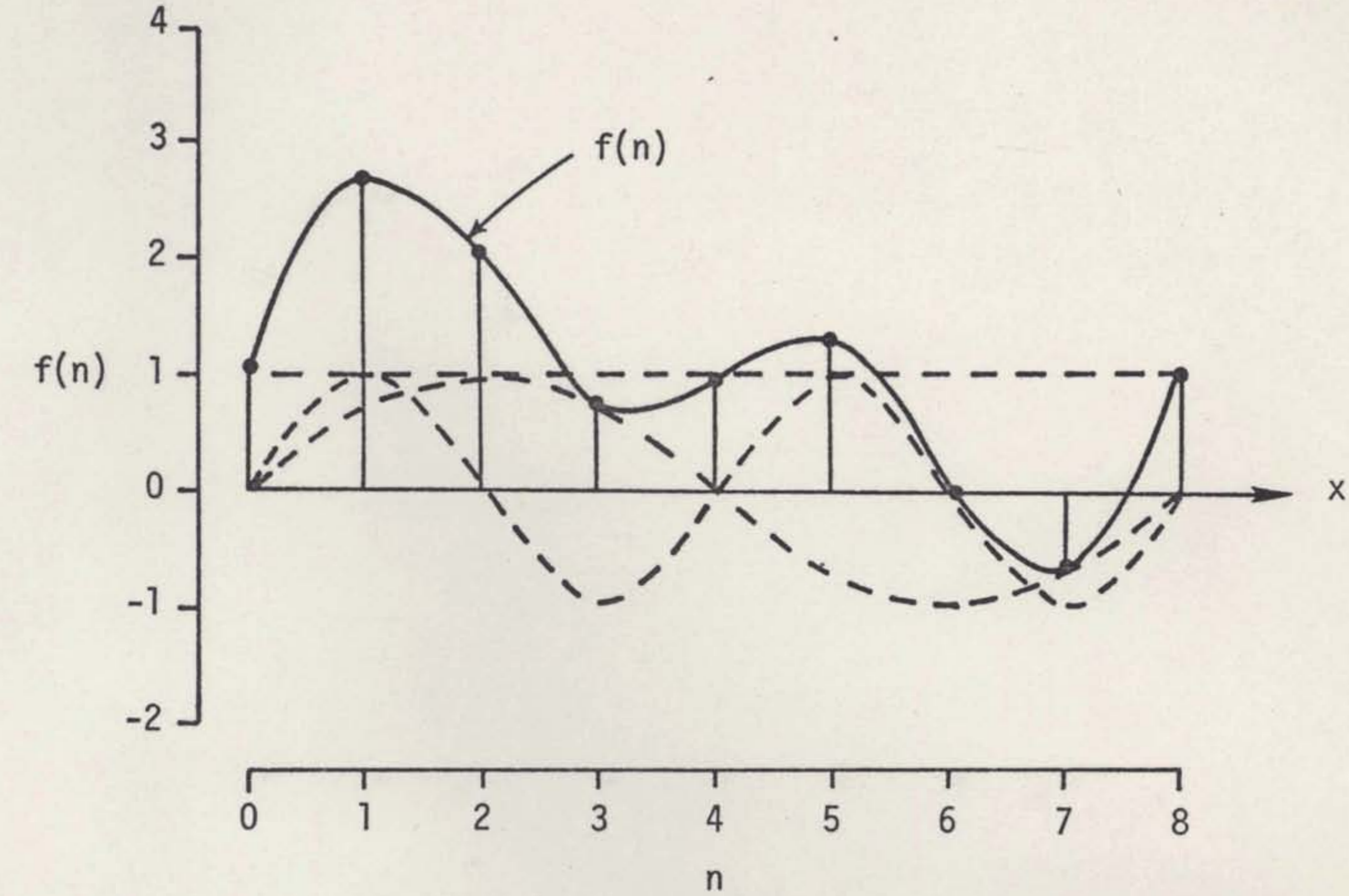
and $f(n) = 1 + \sin\frac{\pi}{4}n + \sin\frac{\pi}{2}n$ (36)

Now, define the two-dimensional function f(m,n) as the sum of the above so that

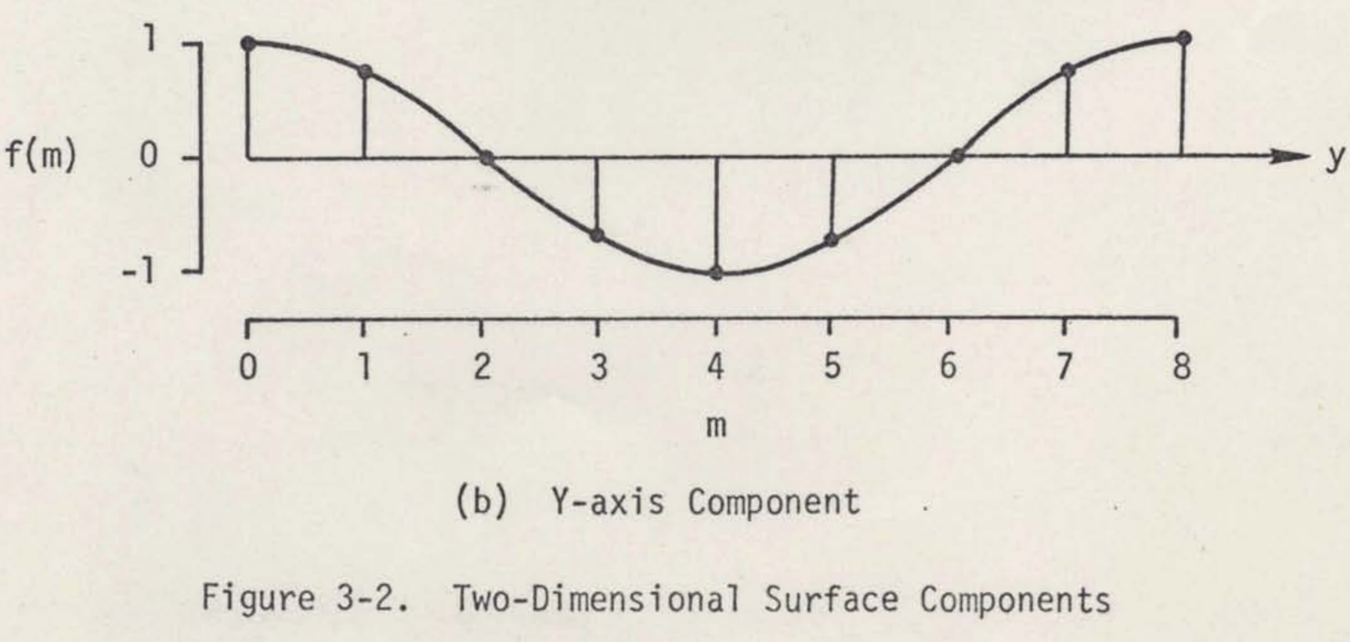
$$f(m,n) = 1 + \sin\frac{\pi n}{4} + \sin\frac{\pi}{2}n + \cos\frac{\pi}{4}m \tag{37}$$

The above function can be represented by an 8x8 array where m and n represent the row and column numbers, respectively. Thus, the matrix $\overline{f}$ in equation (26) becomes

$$\overline{f} = \begin{bmatrix} 2 & 3.707 & 3 & 1.707 & 2 & 2.293 & 1 & 0.293 \\ 1.707 & 3.414 & 2.707 & 1.414 & 1.707 & 2 & 0.707 & 0 \\ 1 & 2.707 & 2 & 0.707 & 1 & 1.293 & 0 & -0.707 \\ 0.293 & 2 & 1.293 & 0 & 0.293 & 0.586 & -0.707 & -1.414 \\ 0 & 1.707 & 1 & -0.293 & 0 & 0.293 & -1 & -1.707 \\ 0.293 & 2 & 1.293 & 0 & 0.293 & 0.586 & -0.707 & -1.414 \\ 1 & 2.707 & 2 & 0.707 & 1 & 1.293 & 0 & -0.707 \\ 1.707 & 3.414 & 2.707 & 1.414 & 1.707 & 2 & 0.707 & 0 \end{bmatrix} \tag{38}$$



(a) X-axis Component



(b) Y-axis Component

Figure 3-2. Two-Dimensional Surface Components

Taking the one-dimensional transform defined by equation (28) where $\overline{W}$ is given by equation (70) of Section 2.0 we get

$$\overline{Z} = \begin{bmatrix} 16 & -j4 & -j4 & 0 & 0 & 0 & +j4 & +j4 \\ 13.657 & -j4 & -j4 & 0 & 0 & 0 & +j4 & +j4 \\ 8 & -j4 & -j4 & 0 & 0 & 0 & +j4 & +j4 \\ 2.343 & -j4 & -j4 & 0 & 0 & 0 & +j4 & +j4 \\ 0 & -j4 & -j4 & 0 & 0 & 0 & +j4 & +j4 \\ 2.343 & -j4 & -j4 & 0 & 0 & 0 & +j4 & +j4 \\ 8 & -j4 & -j4 & 0 & 0 & 0 & +j4 & +j4 \\ 13.657 & -j4 & -j4 & 0 & 0 & 0 & +j4 & +j4 \end{bmatrix} \qquad (39)$$

Finally, transposing the $\overline{Z}$ matrix and performing the operation indicated by equation (33), and again transposing, the TDDFT is given by

$$\overline{F} = \begin{bmatrix} 64 & -j32 & -j32 & 0 & 0 & 0 & +j32 & +j32 \\ 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (40)$$

From the above array the following information about the original array, $\overline{f}$, can be obtained.

1 The dc or constant value, A_0, is given by

$$A_0 = \frac{F_{00}}{N^2} = \frac{64}{64} = 1 \qquad (41)$$

2 Since F_{01} and F_{02} are complex and non-zero this indicates that the variation along the x axis consists of the sum of a first and second harmonic sine wave. The amplitude, C_n, of the n^{th} harmonic is given by

$$C_n = \frac{2}{N^2} |F_{0n}| \tag{42}$$

Therefore, the two sinusoids have magnitudes equal to unity.

3 Since the element F_{10} is real and non-zero this indicates that the variation along the y axis consists of a fundamental cosine wave with an amplitude of unity as determined by equation (42).

4 The non-zero F_{06}, F_{07}, and F_{70} terms are conjugates and correspond to negative frequency values as indicated by equation (50) in Section 2.0.

Using the information obtained in the above steps the original function can be reconstructed as shown in Figure 3-3 where the variations along the x and y axes are as shown in Figure 3-2.

If variations along one axis is dependent on both m and n then some elements not in the first row or column will have non-zero real or complex values. An element having both a real and a complex component indicates that the sinusoidal component has a relative phase shift.

A property of the TDDFT which proves valuable in decreasing computer storage requirements is the fact that elements in the $\overline{F}$ array always appear in complex conjugate pairs and if a shift in the array row and column numbering system is made these pairs are symmetrical about the origin. To demonstrate this property let $\overline{F}$ be an 8x8 array with the normal row and column numbering system as shown in Figure 3-4.

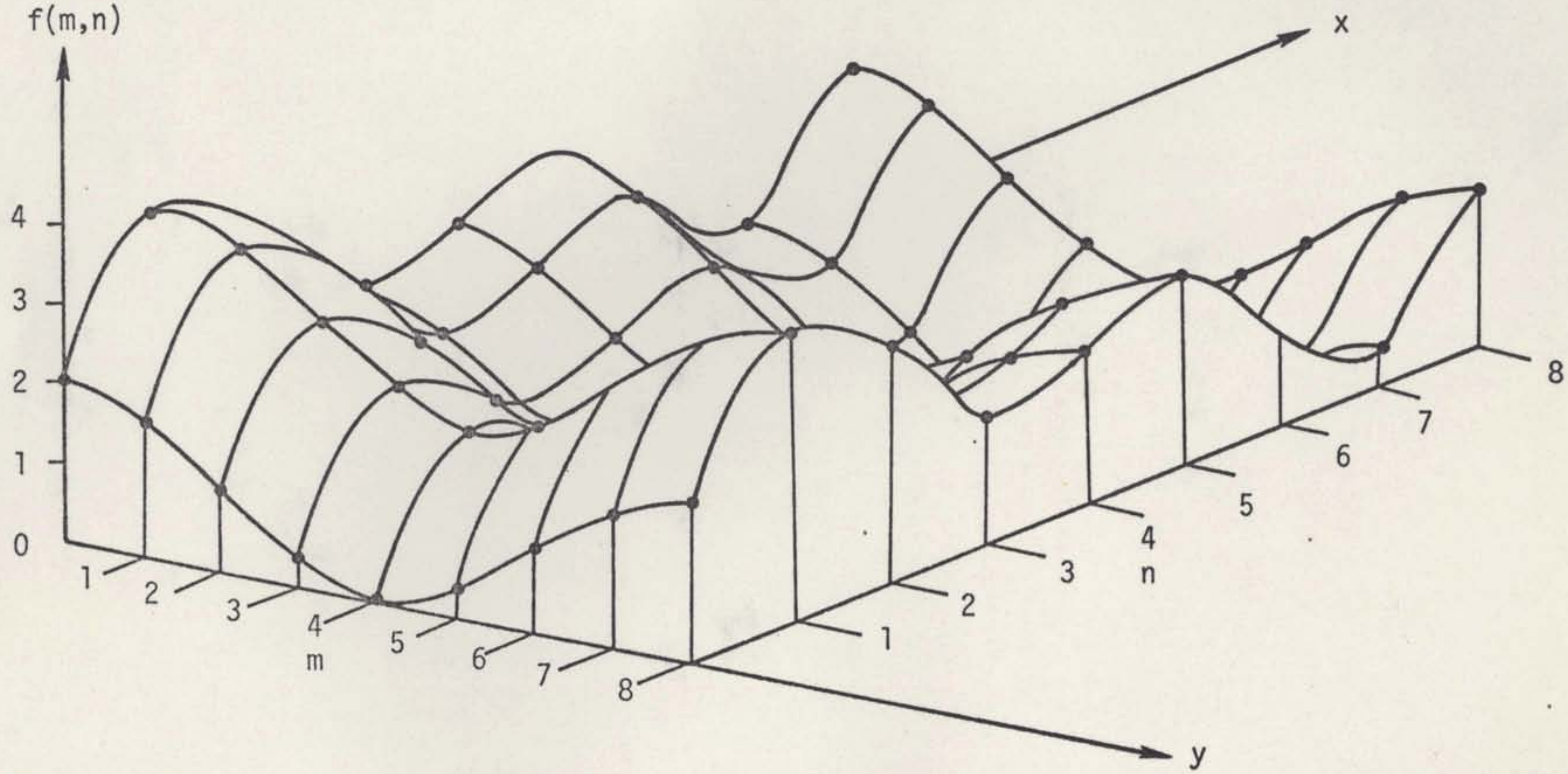


Figure 3-3. Surface Constructed From Three Sinusoidal Waves

Column Number (l)

		0	1	2	3	4	5	6	7
	0:	F_{00}	F_{01}	F_{02}	F_{03}	F_{04}	F_{05}	F_{06}	F_{07}
	1:	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}
	2:	F_{20}	F_{21}	F_{22}	F_{23}	F_{24}	F_{25}	F_{26}	F_{27}
	3:	F_{30}	F_{31}	F_{32}	F_{33}	F_{34}	F_{35}	F_{36}	F_{37}
Row Number (k)	4:	F_{40}	F_{41}	F_{42}	F_{43}	F_{44}	F_{45}	F_{46}	F_{47}
	5:	F_{50}	F_{51}	F_{52}	F_{53}	F_{54}	F_{55}	F_{56}	F_{57}
	6:	F_{60}	F_{61}	F_{62}	F_{63}	F_{64}	F_{65}	F_{66}	F_{67}
	7:	F_{70}	F_{71}	F_{72}	F_{73}	F_{74}	F_{75}	F_{76}	F_{77}

Figure 3-4. F(k,l) Array Numbering System

Since conjugate pairs represent plus and minus frequencies the row and column numbering system of the $\overline{F}$ array can be modified to reflect this fact. Therefore, if the row and column numbers 5, 6, and 7 are changed to -3, -2, and -1, respectively, to reflect negative frequencies the modified form of the array appears as shown in Figure 3-5. This form of the F(k,l) array will result if each element of the f(m,n) array is first multiplied by

$$(-1)^{m+n} \tag{43}$$

before the TDDFT of f(m,n) is performed. Therefore, when the array F(k',l') of Figure 3-5 is formed by rearranging the elements of the F(k,l) array the conjugate pairs are symmetrical about the origin. That is,

$$F(-k',l') = F^*(k',-l') \tag{44}$$

where * denotes the conjugate.

The relationship given by equation (44) requires that only one of each complex conjugate pair in the original F(k,l) array be retained or stored. Thus, computer storage is cut by approximately one half.

Specifically, only the first N/2+1 rows of the F(k,l) array need to be retained. Computer running time is also decreased significantly when the array is to be mathematically altered before performing an inverse transform. The full NxN array is required if the inverse transform is to be performed. The computer subroutine FILL which is discussed in Section 4.0 completes the last N/2-1 rows of the F(k,l) array. The array shown in Figure 3-6 shows the relationship of the conjugate pairs and is identical to the array shown in Figure 3-4.

Modified Column Number (l')

Modified Row Number (k')		<u>-3</u>	<u>-2</u>	<u>-1</u>	<u>0</u>	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>
	-3:	F_{55}	F_{56}	F_{57}	F_{50}	F_{51}	F_{52}	F_{53}	F_{54}
	-2:	F_{65}	F_{66}	F_{67}	F_{60}	F_{61}	F_{62}	F_{63}	F_{64}
	-1:	F_{75}	F_{76}	F_{77}	F_{70}	F_{71}	F_{72}	F_{73}	F_{74}
	0:	F_{05}	F_{06}	F_{07}	F_{00}	F_{01}	F_{02}	F_{03}	F_{04}
	1:	F_{15}	F_{16}	F_{17}	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
	2:	F_{25}	F_{26}	F_{27}	F_{20}	F_{21}	F_{22}	F_{23}	F_{24}
	3:	F_{35}	F_{36}	F_{37}	F_{30}	F_{31}	F_{32}	F_{33}	F_{34}
	4:	F_{45}	F_{46}	F_{47}	F_{40}	F_{41}	F_{42}	F_{43}	F_{44}

Figure 3-5. F(k',l') Array Numbering System

Row Number (k)									
	0:	F_{00}	F_{01}	F_{02}	F_{03}	F_{04}	F_{05}	F_{06}	F_{07}
	1:	F_{10}	.	.	.	.	.	.	F_{17}
	2:	F_{20}	.	.	.	.	.	.	F_{27}
	3:	F_{30}	.	.	.	.	.	.	F_{37}
	4:	F_{40}	F_{41}	F_{42}	F_{43}	F_{44}	F_{45}	F_{46}	F_{47}
	5:	F^*_{30}	F^*_{37}	F^*_{36}	F^*_{35}	F^*_{34}	F^*_{3}	F^*_{32}	F^*_{31}
	6:	F^*_{20}	F^*_{27}	F^*_{26}	F^*_{25}	F^*_{24}	F^*_{23}	F^*_{22}	F^*_{21}
	7:	F^*_{10}	F^*_{17}	F^*_{16}	F^*_{15}	F^*_{14}	F^*_{13}	F^*_{12}	F^*_{11}

* Denotes conjugate.

Figure 3-6. Conjugate Pairs of the F(k,l) Array

4.0 FORTRAN COMPUTER SUBROUTINES

To keep the time spent writing computer programs at a minimum the FORTRAN computer subroutines which were created for this report employ two existing subroutines developed by other parties. The subroutines written specifically for this report use single subscript notation to define the matrix arrays rather than double subscripts so that computer running time is kept at a minimum. Double subscript notation requires that the computer software perform additional multiplication and addition steps to locate an element in an array. Single subscript notation will significantly reduce computer running time if large arrays are processed.

Subroutines TWOFFT, RECORD, PROCOL, and FILL were developed specifically for this report while the subroutines FOUR1 and PLOTB are programs which exist as part of the software of the Martin Marietta Corporation Data Center at Orlando. The FOUR1 subroutine is a version of the Cooley-Turkey Fast Fourier Transform [8] while the PLOTB subroutine is a self-scaling plot routine developed by Martin Marietta Corporation.

Subroutine TWOFFT computes the two-dimensional discrete Fourier transform (TDDFT) of an NxN array as well as the inverse transform. Subroutine TWOFFT requires subroutines FOUR1, RECORD, PROCOL, and PLOTB. The call statement is

```
CALL TWOFFT(N,X,ISIGN,IPR,IPL)
```

where the subroutine arguments are:

N: number of rows or columns of the NxN array.

X: a complex one-dimensional $1xN^2$ array which is transformed and replaced with the transform values

ISIGN: the subroutine will compute the TDDFT if ISIGN equals -1 or will compute the inverse TDDFT if ISIGN equals +1.

IPR: an index which controls the printing of the complex numerical values as the transform is performed.

IPR = 0: print turned off

= 1: print the real part only

= 2: print the imaginary part only

= 3: print the magnitude only

= 4: print the real and imaginary components as well as the magnitude

IPL: an index which allows the numerical values available for printing by the IPR index above to be plotted as a function of the input array index number.

IPL = 0:no plot requested

= 1: plot the real part

= 2: plot the imaginary part

= 3: plot the magnitude

The complete TWOFFT subroutine which performs the two-dimensional discrete Fourier transform is

```
      SUBROUTINE TWOFFT(N,X,ISIGN,IPR,IPL)
      DIMENSION X(1024),Z(1024),IND(1024),Y(1024)
      COMPLEX X,Z
C
C     TAKE FFT OF EACH ROW
C
      NQ=N*N
      DO 1 I=1,N
      K=(I-1)*N+1
      CALL FOUR1(X(K),N,ISIGN)
    1 CONTINUE
      IF(IPR.LE.0.AND.IPL.LE.0) GO TO 4
      PRINT 20
      CALL RECORD(NQ,X,IPR,IPL,103,10)
C
C     TAKE THE MATRIX TRANSPOSE
C
    4 DO 5 I=1,N
      IX=(I-1)*N
      DO5 J=1,N
```

```
      IX=IX+1
      IZ=I+N*(J-1)
      Z(IZ)=X(IX)
    5 CONTINUE
      IF(IPR.LE.0.AND.IPL.LE.0) GO TO 8
      PRINT 21
      CALL RECORD(NQ,Z,3,IPL,103,10)
C
C     TAKE FFT OF ROWS OF TRANSPOSED MATRIX, Z
C
    8 DO 9 I=1,N
      K=(I-1)*N+1
      CALL FOUR1(Z(K),N,ISIGN)
    9 CONTINUE
      IF(IPR.LE.0.AND.IPL.LE.0) GO TO 17
      PRINT 22
      CALL RECORD(NQ,Z,3,IPL,103,10)
C
C     TRANSPOSE MATRIX Z TO GET TWO-DIMENSIONAL FFT
C
   17 DO 18 I=1,N
      IZ=(I-1)*N
      DO 18 J=1,N
      IZ=IZ+1
      IX=I+N*(J-1)
      X(IX)=Z(IZ)
   18 CONTINUE
      XN=1
      IF(ISIGN.GE.1) XN=1./NQ
      DO 10 I=1,NQ
   10 X(I)=X(I)*XN
      IF(IPR.LE.0.AND.IPL.LE.0) RETURN
      PRINT 23
      CALL RECORD(NQ,X,IPR,IPL,103,10)
      RETURN
   20 FORMAT(//50X,'ONE-DIMENSIONAL FFT OF ROWS'/)
   21 FORMAT(//50X,'TRANSPOSE OF ONE-DIMENSIONAL FFT'/)
   22 FORMAT(//50X,'ONE-DIMENSIONAL FFT OF COLUMNS'/)
   23 FORMAT(//50X,'TWO-DIMENSIONAL FFT OF IMAGE'/)
      END
```

Subroutine FOUR1 is called by the subroutine TWOFFT to perform one-dimensional Fast Fourier Transforms on the rows of two-dimensional arrays as discussed in Section 3.0. The call statement is

CALL FOUR1(Y,NE,ISIGN)

where the subroutine arguments are:

NE: number of elements in the row that are to be transformed

Y: a complex one-dimensional 1xNE array which is transformed and replaced with the transform values

ISIGN: the same variable used in the TWOFFT subroutine and specifies the transform direction.

Subroutine RECORD is called by subroutine TWOFFT. The function of subroutine RECORD is to inspect the state of the IPR and IPL arguments and control the plotting and printout of selected variables in the TWOFFT subroutine. The call statement for this subroutine is

CALL RECORD(NQ,X,IPR,IPL,NR,NC)

where the subroutine arguments are:

X: the one-dimensional array which is to be printed or plotted

NQ: total length of the x array

IPR: the same meaning as indicated for the TWOFFT subroutine

IPL: the same meaning as indicated for the TWOFFT subroutine

NC: when the printout of a variable is requested NR specifies the number of rows that are to be printed.

Subroutine RECORD is best explained by citing an example. Suppose the array to be printed has 300 elements and the page is only wide enough to print 10 elements. Then by making NC and NR equal to 10 and 30, respectively, the printout will consist of 30 rows having 10 variables per row. The complete RECORD subroutine is

```
      SUBROUTINE RECORD(N,X,IPR,IPL,NR,NC)
      DIMENSION X(1024),ZR(1024),ZI(1024),ZM(1024),WF(1024)
      COMPLEX X
      DO 1 I=1,N
      WF(I)=I
      ZR(I)=REAL(X(I))
      ZI(I)=AIMAG(X(I))
      ZM(I)=CABS(X(I))
    1 CONTINUE
      IPK=IPR+1
      IF(IPR.NE.4) GO TO (5,2,3,4),IPK
    2 PRINT 20
      CALL PROCOL(NC,NR,ZR)
      IF(IPR.NE.4) GO TO 5
```

```
    3  PRINT 21
       CALL PROCOL(NC,NR,ZI)
       IF(IPR.NE.4) GO TO 5
    4  PRINT 22
       CALL PROCOL(NC,NR,ZM)
    5  IF(IPL.LE.0) RETURN
       IF(IPL.NE.4) GO TO (6.7.8),IPL
    6  CALL PLOTB(1,WF,1,ZR,1,N,1,0,1,0)
       IF(IPL.NE.4) RETURN
    7  CALL PLOTB(1,WF,1,ZI,1,N,1,0,1,0)
       IF(IPL.NE.4) RETURN
    8  CALL PLOTB(1,WF,1,ZM,1,N,1,0,1,0)
       RETURN
   20  FORMAT(/10X,'REAL PART'/)
   21  FORMAT(/10X,'IMAGINARY PART'/)
   22  FORMAT(/10X,'MAGNITUDE'/)
       END
```

Subroutine PROCOL is called by the RECORD subroutine. The function of subroutine PROCOL is to inspect the NC and NR variables and convert the one-dimensional array, x, to an array having NC columns and NR rows. The call statement is

CALL PROCOL(NC,NR,X)

where the subroutine arguments are:

NC: number of columns to be printed.

NR: number of rows to be printed

X: the one-dimensional array that is to be converted to an NRxNC array for printing

The complete PROCOL subroutine is

```
      SUBROUTINE PROCOL(NX,NY,X)
      DIMENSION X(1024),Z(1024)
      N=1024
      NP=MINO(NX,10)
      NMX=NX/NP+1
      DO1 I=1,NY
      NDIF=N-I*NX
      DO2 J=1,NMX
      IP=(J-1)*NP
      IXX=(I-1)*NX+IP
      KI=NX-IP
      KSTOP=MINO(KI,NP)
      IF(KSTOP.LE.0) GO TO 1
      IF(NDIF.LT.0) KSTOP=KSTOP+NDIF
```

```
      DO 3 K=1,KSTOP
      IX=IXX+K
      Z(K)=X(IX)
    3 CONTINUE
      PRINT 10,I,(Z(K),K=1,KSTOP)
    2 CONTINUE
    1 CONTINUE
      RETURN
   10 FORMAT(2X,I3,1X,10F11.4)
      END
```

Subroutine PLOTB is called by the RECORD subroutine and is a self-scaling plot routine which was used solely as a diagnostic device for checking out the TWOFFT program and is not an integral part of the TWOFFT subroutine.

The function of subroutine FILL is to accept the first N/2+1 rows of the F(k,1) array discussed in Section 3.0 and compute the elements that are required to complete the last N/2-1 rows. The subroutine call statement is

```
CALL FILL(NX,X,Y)
```

where the subroutine arguments are:

X: the one-dimensional input array which has the number of elements equal to the number of elements contained in the first N/2+1 rows of the F(k,1) array.

NX: the number of elements contained in the array x.

Y: the one-dimensional output array which contains all elements of the complete F(k,1) array.

The complete FILL subroutine is

```
      SUBROUTINE FILL(NX,X,Y)
      NTOT=NX*NX
      NSTP=NX/2-1
      NRED=NX*(NX-NSTP)
      DIMENSION X(NRED),Y(NTOT)
      COMPLEX X,Y
      NRP=NX-1
      DO 1 N=1,NRED
    1 Y(N)=X(N)
      DO 2 N=1,NSTP
      K=NX*N+1
```

```
      M=NTOT-N*NX+1
    2 Y(M)=CONJG(X(K))
      DO3 N=1,NSTP
      K=NX*N+1
      DO 3 L=1,NRP
      I=K+L
      J=NTOT-NX*(N-1)-L+1
      Y(J)=CONJG(X(I))
    3 CONTINUE
      RETURN
      END
```

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